United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Mary Cartwright

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Sheet 1

Solutions and comments

This programme of the Mentoring Scheme is named after Dame Mary Lucy Cartwright (1900–1998).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Cartwright.html for more information.

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1. This is the first of the Cartwright series of mentoring sheets, named after *Dame Mary Lucy Cartwright*, a mathematician who started the important mathematical field of *chaos theory*, proving some of the important results. What can you find out about her and her areas of work?

SOLUTION

> There's no right answer to this question, and there's lots of interesting areas that can be explored. These are some of the things that I found most interesting about Mary Cartwright.

Dame Mary Lucy Cartwright was a mathematician who studied areas of mathematical analysis and proved some of the very first results concerning chaos theory.

Born in 1900 at Aynho in Northamptonshire, Mary Cartwright was initially most interested in history at school. It was only later on in her school career that she found that she also enjoyed mathematics, the subject that she went on to study at Oxford University.

At Oxford, Mary Cartwright was disappointed after her first two years not to be awarded a first class result at the end of that year. However in her third year she was introduced to mathematical analysis, the study of particular types of functions, which rekindled her mathematical interest. She became the first woman to obtain a first class degree in mathematics from Oxford.

After teaching for a period of four years, in 1928 she returned to Oxford in order to pursue mathematical research. Having obtained her doctorate, she was awarded a fellowship at Girton College Cambridge, where she proved *Cartwright's theorem*, which restricts the size of all functions that are analytic on a disc. For now, take analytic functions to be particularly nice ones, including many of the standard functions you might have met: polynomials, exponentials, trigonometric functions etc.

In 1938 Cartwright went on to undertake work for the Radio Research Board, looking at the crucial wartime technology of radar. Her studies in this area led to the first understanding of chaos theory and the *butterfly effect*. This refers to an imagined butterfly flapping its wings in the Amazon rainforest and causing a tornado some weeks later on the other side of the world. More generally, chaotic situations are ones where small changes in inputs to a system have very large impacts on the resulting development of that system.

Cartwright went on to many other mathematical achievements, including a new and more simple proof that π is irrational. She was honoured in many ways for her work, becoming the first female mathematician to be elected as a Fellow of the Royal Society, the first female president of the London Mathematical Society and then Dame Mary in 1969.

2. Find all real numbers x such that $x - 3 = \sqrt{2x - 3}$.

ANSWER $x = 6$

SOLUTION

If we square the original equation, then we get

$$(x-3)^2 = 2x - 3.$$

Expanding the square yields

$$x^2 - 6x + 9 = 2x - 3.$$

Collecting terms gives

$$x^2 - 8x + 12 = 0.$$

This factorises to give

$$(x - 2)(x - 6) = 0.$$

Therefore we know that $x = 2$ or $x = 6$.

It is now important to check our solutions, particularly as the equation involves a square root.

- If $x = 2$, then $x - 3 = -1$, but $\sqrt{2x - 3} = \sqrt{1} = 1$, so this is not a valid solution.
- If $x = 6$, then $x - 3 = 3$, and $\sqrt{2x - 3} = \sqrt{9} = 3$, so this is a valid solution.

This question is not particularly difficult but it is an example of an equation whose apparent solutions are not all valid. This is because, as in this case, we might have created additional solutions which do not satisfy the original equation.

An alternative to this is to ensure that the logic for all the steps is *reversible*, meaning that we can follow the argument backwards. This then automatically performs the check for us, as we can move through the whole argument in the other direction.

To discuss this we will use the phrase "if and only if". Statement A holds if and only if Statement B holds means both A implies B and B implies A. This makes certain that the solutions we obtain actually do work.

For this example, we cannot reverse our first deduction. If we know that $(x - 3)^2 = 2x - 3$, when we take the positive square root we need to know whether $x - 3 \geq 0$. This means we need to modify our argument.

We know that $x - 3 = \sqrt{2x - 3}$ happens if and only if both

$$(x - 3)^2 = 2x - 3 \text{ and } x \geq 3.$$

Rearranging the first equation can be done in either direction, so this happens if and only if

$$(x - 2)(x - 6) = 0 \text{ and } x \geq 3.$$

Then this happens if and only if one of the following happens:

- $x = 2$ and $x \geq 3$, which is impossible;
- $x = 6$ and $x \geq 3$, which exactly says $x = 6$.

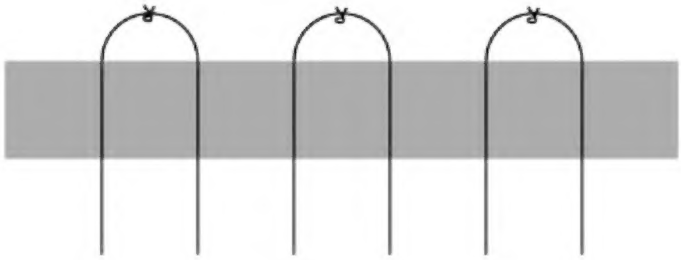
Then, we know that the original equation is true if and only if $x = 6$.

3. Abby holds six pieces of string in her hand with the ends of each of the strings sticking out above and below her hand. Rhiannon ties the upper ends together in pairs, and then does the same with the lower ends. If she ties the pairs randomly, then what is the probability that all six pieces of string will form a single loop?

ANSWER $\frac{8}{15}$

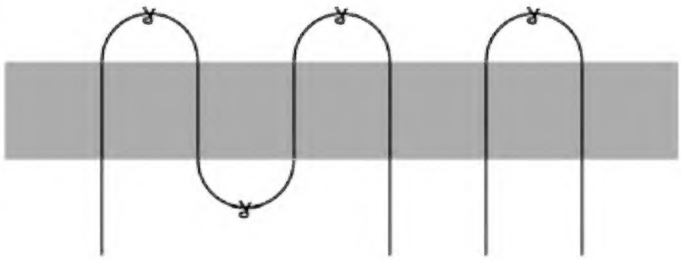
SOLUTION

To start with, Rhiannon joins together three pairs of top ends of the strings, meaning that Abby is holding three longer pieces of string. This yields the following situation:



We then have to consider the knots which occur underneath Abby's hand. It is worth noting that getting exactly one loop is equivalent to making a loop with the final knot only, so we need to work out the probability that no knot before this forms a loop.

Whichever string Rhiannon picks up first, she then has a choice of five others to attach to it. Exactly one of these will form a loop, so she has a probability of $\frac{4}{5}$ of not forming a loop.



There are now four ends remaining below Abby's hand, as in the diagram above. Whichever end Rhiannon selects, there is again exactly one other end which would produce a loop if joined, of the three remaining. This means Rhiannon has a $\frac{2}{3}$ probability of not producing a loop.

At this point, there are only two ends left, so all Rhiannon can do is tie these together to produce a single loop.

The overall probability of producing exactly one loop is $\frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$.

4. Find all pairs of integers x and y such that $xy + 3x - 4y = 29$.

SOLUTION

> It is not immediately obvious how to start this question, as we have got just one equation in two unknowns. It would be good to be able to factorise the left hand side of the equation. Since we have an xy term, we could try and factorise in the form $(x + a)(y + b)$, for suitable values of a and b.

Observe that we have the identity

$$xy + 3x - 4y - 12 \equiv (x - 4)(y + 3).$$

Using this, we can write the original equation as

$$(x-4)(y+3)=17.$$

Then, we need to consider the possible factorisations of 17, since we know that both $x-4$ and $y+3$ are integers. These factorisations are

$$17=1\times 17,\quad 17=17\times 1,\quad 17=-1\times -17,\quad 17=-17\times -1.$$

Comparing each of these factorisations with $(x-4)(y+3)$ gives the following possible solutions:

$$x=5 \text{ and } y=14,\quad x=21 \text{ and } y=-2,\quad x=3 \text{ and } y=-20,\quad x=-13 \text{ and } y=-4.$$

We then need to check that these all do satisfy the original equation. For example, substituting $x=5$ and $y=14$ into the left hand side of the original equation yields

$$5\times 14+3\times 5-4\times 14=70+15-56=29,$$

as required. The other checks are left to the reader.

5. Suppose that

$$x+\frac{1}{x}=5.$$

(a) What is the value of

$$x^2+\frac{1}{x^2}?$$

(b) What is the value of

$$x^5+\frac{1}{x^5}?$$

Answer

(a) 23

(b) 2525

Solution

(a) We know the value of $x+\frac{1}{x}$ and we want to know the value of $x^2+\frac{1}{x^2}$, so it makes sense to try squaring $x+\frac{1}{x}$. This gives:

$$25=\left(x+\frac{1}{x}\right)^2=x^2+2+\frac{1}{x^2}.$$

Subtracting 2 from each side gives:

$$x^2+\frac{1}{x^2}=23.$$

(b) Inspired by the approach used in the first part, we can start by calculating some of the powers of $x+\frac{1}{x}$.

Since we have coefficients of x^5 and $\frac{1}{x^5}$, we can start by calculating

$$\left(x+\frac{1}{x}\right)^5 = x^5+5x^3+10x+\frac{10}{x}+\frac{5}{x^3}+\frac{1}{x^5}.$$

This means there are extra terms with coefficients of x^3 and $\frac{1}{x^3}$, so we compute

$$\left(x+\frac{1}{x}\right)^3 = x^3+3x+\frac{3}{x}+\frac{1}{x^3}.$$

Then, using these relationships, we can calculate the following:

$$\begin{aligned}
x^5+\frac{1}{x^5} &= \left(x+\frac{1}{x}\right)^5 - 5x^3 - 10x - \frac{10}{x} - \frac{5}{x^3} \\
&= \left(x+\frac{1}{x}\right)^5 - 5\left(x+\frac{1}{x}\right)^3 + 5x + \frac{5}{x} \\
&= \left(x+\frac{1}{x}\right)^5 - 5\left(x+\frac{1}{x}\right)^3 + 5\left(x+\frac{1}{x}\right).
\end{aligned}$$

Substituting 5 for $x+\frac{1}{x}$ gives:

$$x^5+\frac{1}{x^5} = 5^5 - 5\times 5^3 + 5\times 5 = 3125 - 625 + 25 = 2525.$$

In the question, it is worth noticing that all of the expressions have the same coefficients for the powers of x as they do for $\frac{1}{x}$. This makes them all *symmetrical*, which allows us to compute the values of them from our simple symmetrical expression $x+\frac{1}{x}$.

6. (a) Prove that $a^2+b^2 \geq 2ab$ for all real numbers a and b. When does equality occur?

(b) Prove that $a^2+b^2+c^2 \geq ab+bc+ca$ for all real numbers a, b and c. When does equality occur?

Solution

I find it often helps to think about this sort of question in reverse: starting with the thing we want to prove, and then trying to get back to something that we know is true. When we have done this, we then need to make sure we can do all the steps backwards - it is this backwards part that becomes the actual solution.

So, if I wanted to prove that $a^2+b^2 \geq 2ab$, then I would want to subtract $2ab$ from each side, meaning I would want to prove that $a^2-2ab+b^2 \geq 0$.

The left hand side of this inequality factorises as $(a-b)^2$, which we know is at least 0. Let us reverse this argument in what follows!

(a) Since squares are always non-negative, we know that

$$(a-b)^2 \geq 0.$$

Then, we can expand the left hand side to get

$$a^2 - 2ab + b^2 \geq 0.$$

Then, adding $2ab$ to each side gives, as required,

$$a^2 + b^2 \geq 2ab.$$

In order to get equality in this, we must have equality throughout the argument above, so $a - b$ must equal 0. This means we can only get equality if $a = b$. It can be checked that this does indeed give equality, since $a^2 + a^2 = 2a^2$.

(b) Dividing the inequality that we obtained in the first part of this question by 2 gives $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$. We can then repeat this for a and c and for b and c, giving:

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2, \quad ac \leq \frac{1}{2}a^2 + \frac{1}{2}c^2, \quad bc \leq \frac{1}{2}b^2 + \frac{1}{2}c^2.$$

Then these three equations can be added together to give, as required,

$$ab + bc + ca \leq a^2 + b^2 + c^2.$$

Moreover, to get equality overall, each of the three inequalities we used above must actually be equalities. From the first part, this means that $a = b = c$. This clearly then does give equality, as both sides are equal to $3a^2$.

7. Find all integers n such that $n^2 - 7n + 10$ is divisible by $n - 3$.

Answer 1, 2, 4, 5

Solution

> The key fact that we need for this question is that if a is divisible by b, then we can subtract (or add) any multiple of b to a and still get another multiple of b.
>
> To see that this is true, argue as follows. As a is a multiple of b, we can write $a = mb$. Then, for any integer x, $a - xb = mb - xb = (m - x)b$ is a multiple of b.

Suppose that $n^2 - 7n + 10$ is divisible by $n - 3$. Clearly $n(n-3) = n^2 - 3n$ is divisible by $n - 3$. This means $(n^2 - 7n + 10) - (n^2 - 3n) = 10 - 4n$ is divisible by $n - 3$.

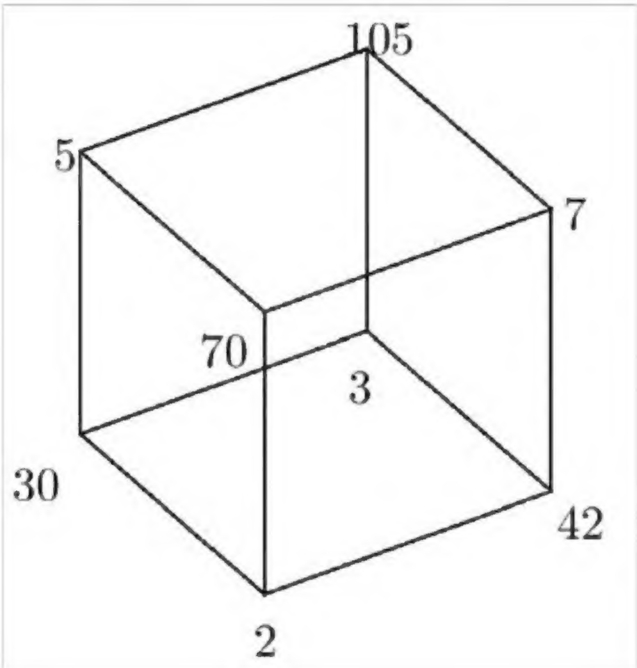
We also know $4(n-3) = 4n - 12$ is divisible by $n - 3$. We can then add this to $10 - 4n$, giving -2. This must still be divisible by $n - 3$ but the only factors of -2 are -2, -1, 1 and 2.

This means the only possible values of n are 1, 2, 4 and 5. All that remains is to check that these are all valid solutions, which can be done with the following table.

n	$n-3$	$n^2-7n+10$	Divisible?
1	-2	4	Yes
2	-1	0	Yes
4	1	-2	Yes
5	2	0	Yes

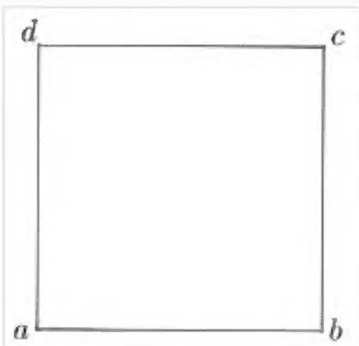
8. Is it possible to place positive integers at each of the vertices of a cube in such a way that, for each pair of numbers connected by an edge, one is divisible by the other and for every pair of numbers not connected by an edge, neither number is divisible by the other number? *(Note that the condition in the question requires that no two unconnected vertices should have one number divisible by the other.)*

ANSWER Yes; an example is shown in the diagram below.



SOLUTION

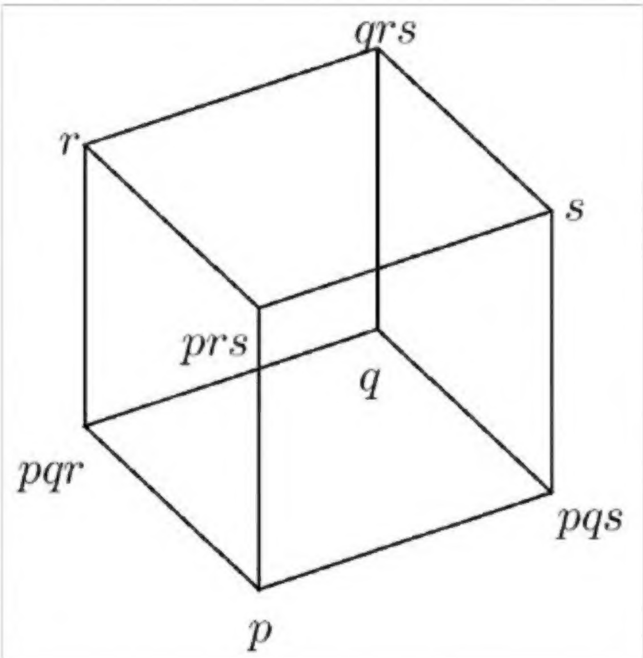
Let us start by considering the two-dimensional case of a square. The following diagram shows an appropriately labelled square.



Now, if $b = ka$ and $c = lb$, then $c = kla$. Since a and c are not connected, this cannot happen, so each vertex must either be a factor of all adjacent vertices, or a multiple of all adjacent vertices.

It then makes sense to label those vertices which provide factors by prime numbers. We can make a vertex label divisible by such a prime by including it as a factor in the label. Any unconnected vertex needs a coprime label and so we should not include in its label the prime under consideration.

This means that we can make four of the vertices of the cubes be primes, say p, q, r and s, and the other vertices the product of their neighbours, as in the diagram below. Each prime vertex divides all its vertices and no others.



To get some actual numbers, you can choose your favourite primes for p, q, r and s. Choosing $p = 2$, $q = 3$, $r = 5$ and $s = 7$ gives the example at the top of the solution.

9. Solve the equation $x^4 - x^3 - 4x^2 - x + 1 = 0$.

ANSWER $x = \frac{3+\sqrt{5}}{2}$, $x = \frac{3-\sqrt{5}}{2}$ or $x = -1$

SOLUTION

> It does not immediately seem obvious how to start this question but one thing that you might notice is that the coefficients of successive powers go 1, −1, −4, −1, 1, which is a symmetrical pattern. This is close to what the expressions in Question 5 looked like, so we can try to write them in a similar form.

The value of x is clearly not zero, as the equation would then be false, so we can divide the whole equation by x^2 to get

$$x^2 - x - 4 - \frac{1}{x} + \frac{1}{x^2} = 0.$$

Since $\left(x+\frac{1}{x}\right)^2 = x^2+2+\frac{1}{x^2}$, this equation can be rewritten as

$$\left(x+\frac{1}{x}\right)^2 - \left(x+\frac{1}{x}\right) - 6 = 0.$$

This is now a quadratic equation in terms of $x+\frac{1}{x}$. This equation factorises as:

$$\left(\left(x+\frac{1}{x}\right)-3\right)\left(\left(x+\frac{1}{x}\right)+2\right) = 0.$$

We can multiply again by x^2 and rewrite this as

$$\left(x^2-3x+1\right)\left(x^2+2x+1\right) = 0. \tag{1}$$

Then, using the quadratic formula to solve these two equations, we get the solutions $x=\frac{3+\sqrt{5}}{2}$, $x=\frac{3-\sqrt{5}}{2}$ or $x=-1$.

To check that these are valid solutions, we need only substitute them into (1), since the left hand side of this expands to give the original equation.

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Sheet 2

Solutions and comments

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1. (a) i. How many ways are there for three people (Alice, Becky and Colin) to arrange themselves into a line? Suppose they are joined by Debayan: how many ways are there now?

ii. How many ways are there for n people to arrange themselves into a line?

You might find it useful to use the notation $n! = 1 \times 2 \times \cdots \times n$ ("n factorial") to express your answer.

(b) i. Mo, Nadya, Olivia and Peter all sign up to be in the doubles badminton team. In how many ways can I choose a team of two from this set of four people?

ii. If there are n people signed up and I need to choose k of them for a team, then how many ways are there of doing this?

The number of ways of choosing k people from a set of n (without replacement) is often written as $\binom{n}{k}$ or ${}^nC_k$.

ANSWER

(a) i. 6, 24

ii. $n!$

(b) i. 6

ii. $\frac{n!}{k!(n-k)!}$

SOLUTION

(a) i. We can systematically list the different options that are available for the line of people. If Alice is first, then there are two options: either Becky is next and Colin last, or Colin is next then Becky is last. This gives the options ABC and ACB. If Becky is first, we get the options BAC and BCA, depending on who is next. If Colin is first then the available options are CAB and CBA. This gives a total of 6 options.

With Debayan also, there are four choices for the first person in the queue. No matter which of these you choose, there are three people available for the second slot. Then there are two people available for the third slot, and just one person left for the last slot. This means there are $4 \times 3 \times 2 \times 1 = 4! = 24$ possibilities.

> $4!$, read as "4 factorial", means $4\times3\times2\times1$. In general, $n! = n\times(n-1)\times(n-2)\times\ldots\times2\times1$.

ii. There are n options for the first person in the line. There are then $n-1$ people left for the next position. There are $n-2$ people available for the next position. Keeping on doing this yields a total of $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = n!$ options.

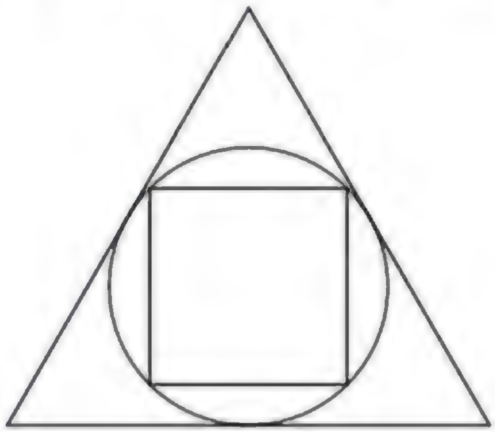
(b) i. There are four possible choices for the first person in the team. There are then three choices for the second person in the team, so this gives $3 \times 4 = 12$ options. However, we have counted each of the pairs twice: for example, Mo and Nadya appear as MN and NM. This means the true number of teams is $12 \div 2 = 6$.

> This is an example of *multiple counting*, where we counted each of the possible teams twice, but then compensated for this by dividing by the number of times we had counted each team. This idea is very useful in part (b) of this question.

ii. Imagine choosing the team by lining all n people up in a queue and then picking the first k in that queue to form the team. From part (a), there are $n!$ different queues of this type.

We now need to work out how many times we have counted the same team. The team is affected only by the first k people in the queue, so we can reorder these k people, and there are $k!$ ways of doing this. We can also reorder the other $n - k$ people without affecting who is selected. There are $(n - k)!$ ways of doing this. This means the total number of rearrangements that keep the team the same is $\frac{n!}{k!(n-k)!}$.

2. A square is inscribed in (drawn inside, just touching) a circle, which is then inscribed in an equilateral triangle. The side length of the equilateral triangle is 60cm.



(a) What is the radius of the circle?

(b) What is the area of the square?

You should be able to answer this question using *exact* values, such as square roots, rather than approximations using your calculator.

ANSWER

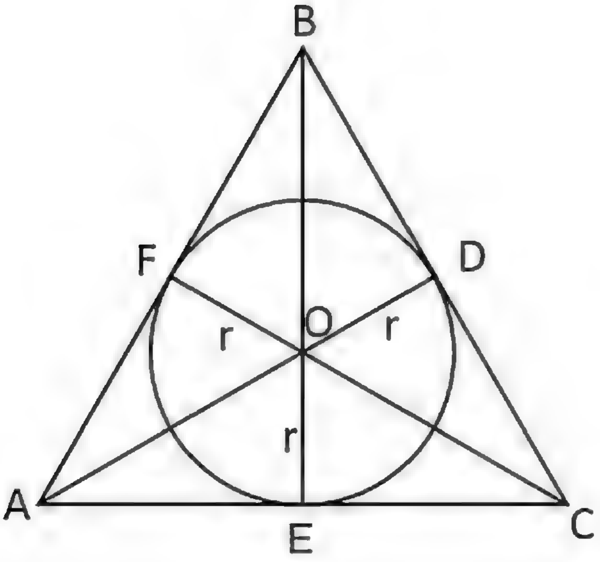
(a) $10\sqrt{3}$cm

(b) 600cm^2

SOLUTION

(a) Let O be the centre of the circle. This must be at the point where the perpendiculars to each side from the opposite vertices meet. (If you are not sure about this, revise what you have learned about triangle centres. If necessary, ask your mentor for help.)

The diagram below illustrates the situation.



We want to calculate the height BE of the equilateral triangle. Using Pythagoras' theorem on triangle ABE gives

$$BE = \sqrt{AB^2 - AE^2} = \sqrt{60^2 - 30^2} = \sqrt{2700} = 30\sqrt{3}.$$

By symmetry, this is also the length of AD and CF.

Then, writing r for the radius of the circle and using Pythagoras' theorem on triangle AOE, we have

$$\left(30\sqrt{3} - r\right)^2 = AO^2 = AE^2 + OE^2 = 30^2 + r^2.$$

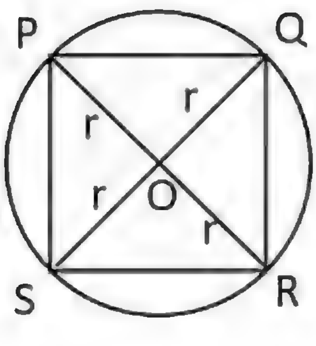
Expanding the left hand side of this gives

$$2700 - 60\sqrt{3}r + r^2 = 900 + r^2.$$

Now subtract r^2 from both sides, collect like terms and divide by $60\sqrt{3}$ to obtain

$$r = 10\sqrt{3}\text{cm}.$$

(b) The diagonals of the square must meet at the centre of the circle, since this point is the same distance from all four vertices.



Since the diagonals meet at right angles, we know $\angle QOR = 90^\circ$ and so we can apply Pythagoras' theorem to triangle QOR. This gives

$$QR^2 = OQ^2 + OR^2 = r^2 + r^2 = 2r^2 = 2 \times \left(10\sqrt{3}\right)^2 = 600.$$

But, QR^2 is exactly the area of the square, so this gives the area as 600cm^2.

3. (a) Jasmine has a line of ten boxes in front of her, all of which are initially closed.

- In the first round she opens all the boxes.
- In the second round, she shuts every second box (namely the second, fourth, sixth, eighth and tenth).
- In the third round, she alters the state (closes if open, opens if closed) of every third box, starting with the third.

...

- In the n^{th} round she changes the state of every n^{th} box.

She does this for 10 rounds. Which boxes are open at the end of the process?

(b) Suppose now that she has 100 boxes in front of her, and continues for 100 rounds. Which boxes are open at the end of the process?

Answer

(a) 1, 4, 9

(b) All the square numbers.

Solution

(a) We can fill out a table to show when each box changes state as follows.

	Box 1	**Box 2**	**Box 3**	**Box 4**	**Box 5**	**Box 6**	**Box 7**	**Box 8**	**Box 9**	**Box 10**
Round 1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Round 2		✓		✓		✓		✓		✓
Round 3			✓			✓			✓	
Round 4				✓				✓		
Round 5					✓					✓
Round 6						✓				
Round 7							✓			
Round 8								✓		
Round 9									✓	
Round 10										✓
Total	1	2	2	3	2	4	2	4	3	4

For a box to be open at the end, it must have been open and shut an odd number of times. This means that the boxes that remain open at the end are 1, 4 and 9.

(b) Consider round r. In this round, every r^{th} box has its state changed, so exactly those boxes that have r as a factor are changed. This means any box, over the course of all the rounds, is changed once for every one of its factors. Therefore, for the box to be open at the end, it must have an odd number of factors.

Since factors come in pairs, the number of factors that a number has, in general, will be even. An exception to this rule occurs when one of the pairs consists of two equal numbers. In other words, the number is a square. In this case all the other factors will pair up and the square root is left to make the number of factors odd. We conclude that the boxes corresponding to square numbers are the ones that will be left open.

4. (a) Find all five-digit numbers which are reversed when multiplied by 4.

You might find it useful to think of this as a column product, in the form:

$$\begin{array}{cccccc} & A & B & C & D & E \\ \times & & & & & 4 \\ \hline & E & D & C & B & A \\ \hline \end{array}$$

Remember that the digits *can* be the same!

(b) Find all five-digit numbers which are reversed when multiplied by 9.

(c) Are there any five-digit numbers which are reversed when multiplied by 8?

ANSWER

(a) 21978

(b) 10989

(c) No possible solutions.

SOLUTION

(a) Writing the calculation in columns, we have the following.

$$\begin{array}{cccccc} & A & B & C & D & E \\ \times & & & & & 4 \\ \hline & E & D & C & B & A \\ \hline \end{array}$$

Then, looking at the ten-thousands column, we know that $4A \leq E$. Since E is a digit, it is at most 9, so A is either 1 or 2. Looking at the units column, we know that $EDCBA$ is a multiple of 4, so it is even, which means A is even. Putting these first two results together means $A = 2$. Note that A is not zero as it is a first digit.

Then, $4E$ must end in 2. The only digits which achieve this are 3 and 8. But, we know $E \geq 4A = 8$, so $E = 8$. The calculation becomes

$$\begin{array}{cccccc} & 2 & B & C & D & 8 \\ \times & & & & & 4 \\ \hline & 8 & D & C & B & 2 \\ \hline \end{array}$$

Now, the ten-thousands column has no carry into it, so we know that $4B \leq D \leq 9$, so $B \leq 2$. Looking at the tens column, $4D + 3$ has last digit B, as the 3 is carried. As $4D$ is even, $4D + 3$ is odd, and so B is odd. As $B \leq 2$, this means $B = 1$.

Since $4B \leq D$, this tells us that $D \geq 4$. We also know that $4D + 3$ has final digit 1, so $4D$ has final digit 8. Consequently $D = 2$ or $D = 7$. Together, these facts tell us that $D = 7$. We have

$$\begin{array}{cccccc} & 2 & 1 & C & 7 & 8 \\ \times & & & & & 4 \\ \hline & 8 & 7 & C & 1 & 2 \\ \hline \end{array}$$

Looking at the carries around the hundreds column, this gives $4C + 3 = C + 30$. Collecting like terms gives $3C = 27$, so $C = 9$.

We need to check that this works and indeed 21978×4 equals 87912. Thus the solution found is valid.

(b) The multiplication is

$$\begin{array}{cccccc} & A & B & C & D & E \\ \times & & & & & 9 \\ \hline & E & D & C & B & A \\ \hline \end{array}$$

We know A is not zero, as it is the first digit. Looking at the ten-thousands column, $9A \leq E \leq 9$, so $A \leq 1$. As A is non-zero, this tells us that $A = 1$ and $E = 9$.

The sum becomes:

$$\begin{array}{cccccc} & 1 & B & C & D & 9 \\ \times & & & & & 9 \\ \hline & 9 & D & C & B & 1 \\ \hline \end{array}$$

The same argument in the thousands column tells us that $B \leq 1$, so $B = 0$ or $B = 1$.

- If $B = 1$, then $9D + 8$ has the last digit 1 from the tens column, so $9D$ has last digit 3 and so $D = 7$.
- If $B = 0$, then $9D + 8$ has the last digit 0 from the tens column, so $9D$ has last digit 2 and so $D = 8$.

We know $9B \leq D$, so this means we must have $B = 0$ and $D = 8$. The calculation is then

$$\begin{array}{cccccc} & 1 & 0 & C & 8 & 9 \\ \times & & & & & 9 \\ \hline & 9 & 8 & C & 0 & 1 \\ \hline \end{array}$$

Here the hundreds column tells us that $9C + 8 = 80 + C$, so $8C = 72$ and $C = 9$. Again this solution is valid, as $10989 \times 9 = 98901$.

(c) This time the attempted multiplication is

$$\begin{array}{cccccc} & A & B & C & D & E \\ \times & & & & & 8 \\ \hline & E & D & C & B & A \\ \hline \end{array}$$

Then, since A is non-zero, $A \geq 1$. But from the ten-thousands column, $8A \leq E \leq 9$, as E is a digit, so $A = 1$. This means $EDCBA$ ends in a 1, so is odd. Since it should be a multiple of 8, this cannot occur, so there are no solutions.

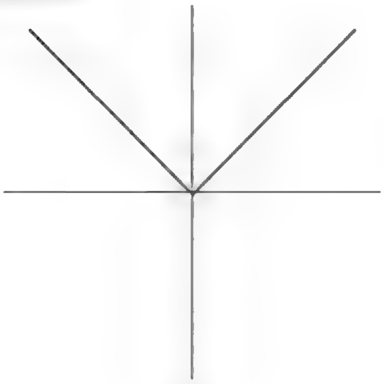
5. The expression $|z|$, read as the *modulus of* z, takes the value z if $z \geq 0$; otherwise it takes the value $-z$.

(a) Sketch the graph of $y = |x|$.

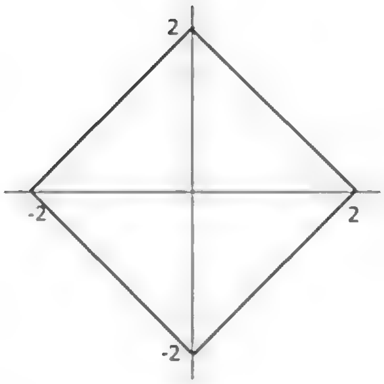
(b) Plot the graph of the points where $|x| + |y| = 2$. What is the area it encloses?

(c) What is the area enclosed by the graph of $|y - x| + |y| = 2$?

ANSWER



(a)



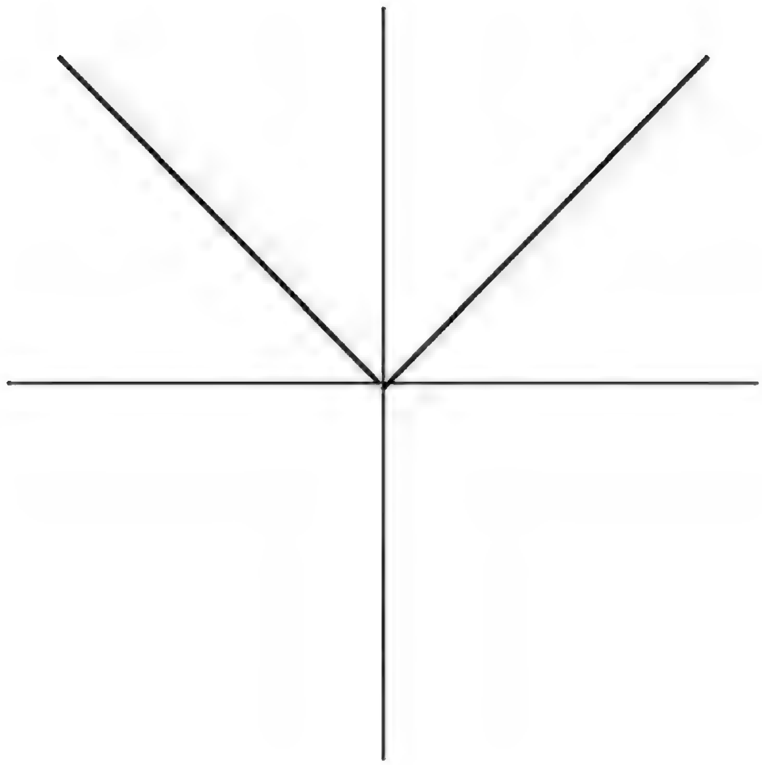
(b)

Area is 8

(c) 8

SOLUTION

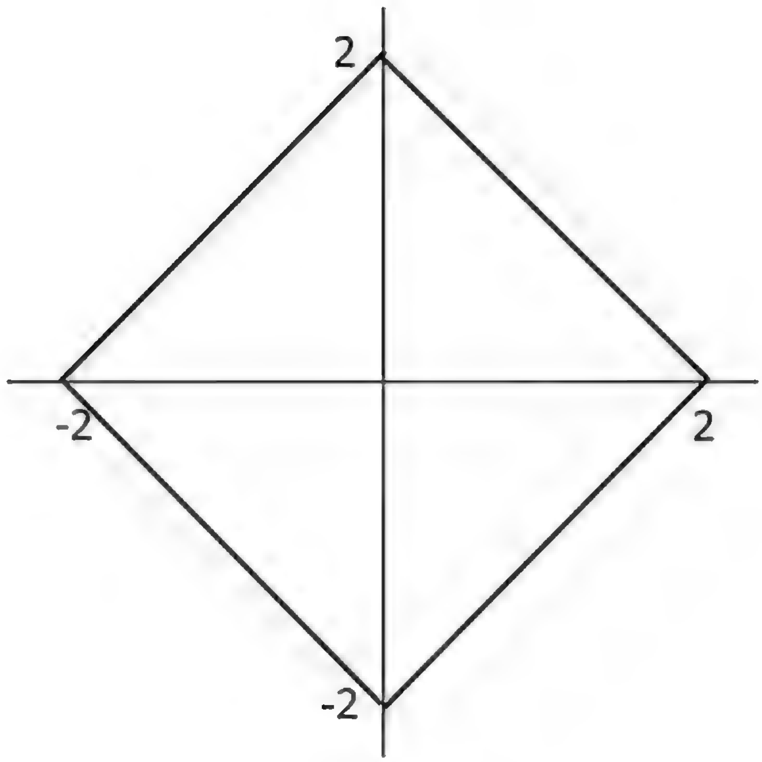
(a) Consider separately the cases where $x \leq 0$ and $x \geq 0$. For $x \leq 0$, we draw the line $y = -x$. For $x \geq 0$, we draw the line $y = x$. This produces the following graph.



(b) There are four different cases to consider.

- $x \geq 0$ and $y \geq 0$. Then the line is $x + y = 2$, which rearranges to $y = -x + 2$.
- $x \geq 0$ and $y \leq 0$. Then the line is $x - y = 2$, which rearranges to $y = x - 2$.
- $x \leq 0$ and $y \geq 0$. Then the line is $-x + y = 2$, which rearranges to $y = x + 2$.
- $x \leq 0$ and $y \leq 0$. Then the line is $-x - y = 2$, which rearranges to $y = -x - 2$.

Plotting these lines in these ranges gives the diagram below.



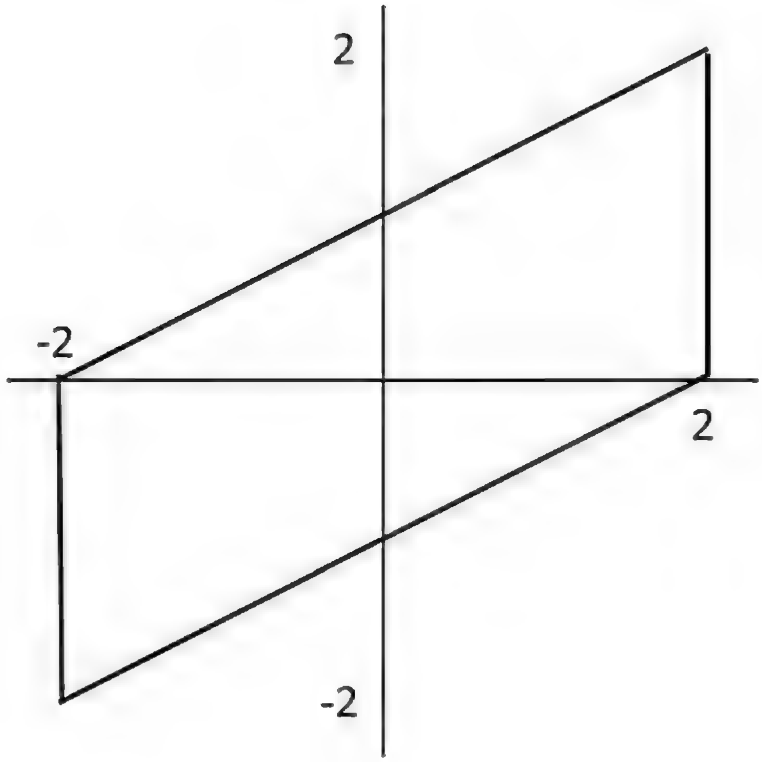
This shape is composed of four right angled triangles, one in each quadrant. Each has width 2 and

height 2, so each has area $\frac{1}{2} \times 2 \times 2 = 2$. Thus the total area is 8.

(c) Again there are four cases to consider

- $y \geq 0$ and $y \geq x$. Then the line is $y - x + y = 2$, which rearranges to $y = \frac{1}{2}x + 1$.
- $y \geq 0$ and $y \leq x$. Then the line is $x - y + y = 2$, which rearranges to $x = 2$.
- $y \leq 0$ and $y \geq x$. Then the line is $y - x - y = 2$, which rearranges to $x = -2$.
- $y \leq 0$ and $y \leq x$. Then the line is $x - y - y = 2$, which rearranges to $y = \frac{1}{2}x - 1$.

This gives the figure below.



This shape is a parallelogram. The base (vertical) has length 2, and the perpendicular height is 4. Therefore the area is $2 \times 4 = 8$.

6. Mr and Mrs Jones are at a party with four other couples. As they all arrived, some pairs of people shook hands with each other. No pair of people shook hands more than once, and no-one shook hands with their partner. When Mr Jones talked to all the other attendees, he found that they had shaken hands with 0, 1, 2, 3, 4, 5, 6, 7 and 8 other people respectively. How many people had Mrs Jones shaken hands with?

ANSWER 4

SOLUTION

No one shook hands with themselves or their partner, so the most hands anyone shook was eight. Label the people apart from Mr Jones by the number of people's hands they shook.

The table below shows which people have shaken hands with which others.

Hands shaken	0	1	2	3	4	5	6	7	8	x
0	×									
1		×								
2			×							
3				×						
4					×					
5						×				
6							×			
7								×		
8									×	
x										×

In this table we have used the fact that no-one shook their own hand. We also know that the person who shook zero hands cannot have shaken with anyone. We can therefore fill that row and column with crosses, as in the next table.

Hands shaken	0	1	2	3	4	5	6	7	8	x
0	×	×	×	×	×	×	×	×	×	×
1	×	×								
2	×		×							
3	×			×						
4	×				×					
5	×					×				
6	×						×			
7	×							×		
8	×								×	
x	×									×

Now consider the person who shook eight hands. There are only eight available spaces, so these must all be ticks.

Hands shaken	0	1	2	3	4	5	6	7	8	x
0	×	×	×	×	×	×	×	×	×	×
1	×	×							✓	
2	×		×						✓	
3	×			×					✓	
4	×				×				✓	
5	×					×			✓	
6	×						×		✓	
7	×							×	✓	
8	×	✓	✓	✓	✓	✓	✓	✓	×	✓
x	×								✓	×

This means we know exactly what the person who shook one hand did, so we can remove all the other options. This then leaves only seven possibilities for the person who shook seven hands, so these must all be ticks. This gives the next table.

Hands shaken	0	1	2	3	4	5	6	7	8	x
0	×	×	×	×	×	×	×	×	×	×
1	×	×	×	×	×	×	×	×	✓	×
2	×	×	×					✓	✓	
3	×	×		×				✓	✓	
4	×	×			×			✓	✓	
5	×	×				×		✓	✓	
6	×	×					×	✓	✓	
7	×	×	✓	✓	✓	✓	✓	×	✓	✓
8	×	✓	✓	✓	✓	✓	✓	✓	×	✓
x	×	×						✓	✓	×

Then we can repeat these steps. We know exactly who the person who shakes hands with two people shakes with, so all the others are crosses. This leaves only six possibilities for the person with six handshakes, so they shake with these.

This completes the three shakes for the person with three, so they shake with no-one else. This leaves only five options for the person with five shakes, so they shake all of these. This then commits all the shakes for the person with four, so they shake with no-one else.

The final table as follows.

Hands shaken	0	1	2	3	4	5	6	7	8	x
0	×	×	×	×	×	×	×	×	×	×
1	×	×	×	×	×	×	×	×	✓	×
2	×	×	×	×	×	×	×	✓	✓	×
3	×	×	×	×	×	×	✓	✓	✓	×
4	×	×	×	×	×	✓	✓	✓	✓	×
5	×	×	×	×	✓	×	✓	✓	✓	✓
6	×	×	×	✓	✓	✓	×	✓	✓	✓
7	×	×	✓	✓	✓	✓	✓	×	✓	✓
8	×	✓	✓	✓	✓	✓	✓	✓	×	✓
x	×	×	×	×	×	✓	✓	✓	✓	×

At this point we know that $x = 4$, so Mr Jones shook hands with four people. We now need to work out who his partner is - we can do this by pairing the people up, remembering that no-one shakes hands with their partner.

The person with eight handshakes only does not shake with the person with 0, so these are partners. The person with 7 is left then with only the person with 1, so these pair. In the same way, we pair up 6 with 2 and 5 with 3. This leaves the two people with 4 together, one of whom is Mr Jones. The other is Mrs Jones, so she shook four hands.

7. Four dice, coloured red, blue, yellow and green, are rolled. In how many different ways can the product of the numbers rolled equal 36? (Here, for example, a red 4 is considered to be different from a blue 4.)

ANSWER 48

Solution

> There are many different ways that you can solve this question, all with various merits. The solution presented here makes use of the ideas from Question 1.

Let us first consider the case where we do not worry about the order. We will look at the largest number used. Since 5 is not a factor of 36, we can never use this. Also, we must have a factor of 3 somewhere, so this is at least 3. It is worth noticing that the prime factorisation of 36 is $2^2 \times 3^2$.

- If the largest number is 6, then the other numbers must multiply to six. One of these must have a factor of 3, which means using either 3 or 6 also. The leftover two numbers must then multiply to 2 or 1, and there is only one way to do each of these. This gives $6 \times 6 \times 1 \times 1$ and $6 \times 3 \times 2 \times 1$ as the only options.
- If the largest number is 4, then the other three numbers must multiply to give 9. The only way to arrange this is $3 \times 3 \times 1$, giving $4 \times 3 \times 3 \times 1$.
- If the largest number is 3, then the only allowed numbers are 1, 2 and 3. Since we have two factors of 2 and two factors of 3, the only option is $3 \times 3 \times 2 \times 2$.

Now we need to consider the order of these numbers on the dice. If we imagine lining the dice up in the order red, green, yellow and blue, then this is the same as finding the number of orders of the given numbers. We can do this separately for each case:

- If the numbers are 6, 6, 1 and 1, then to make each order corresponds to selecting two of the four dice to be sixes. This can be done, using question 1, in $\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2\times 2} = 6$ ways.
- If the numbers are 6, 3, 2 and 1, then all the numbers are different. This means that the total number of orders is, from question 1, $4! = 24$.
- If the numbers are 4, 3, 3 and 1, then there are $\binom{4}{2} = 24$ ways of choosing which two of the numbers are threes. Then, the other two numbers can be in either order, so there are a total of $2 \times 6 = 12$ ways.
- If the numbers are 3, 3, 2 and 2, then there are $\binom{4}{2} = 6$ ways to choose which of the dice show threes, with the others showing twos.

Therefore, the total number of ways of obtaining four dice with a product of 36 is $6 + 24 + 12 + 6 = 48$.

8. (a) Can 101 be written as the sum of at least two consecutive positive integers? What about 102? What about 100?

(b) What integers is it possible to write as the sum of at least two consecutive positive integers? Are there any integers that are impossible to write as the sum of at least two consecutive positive integers?

Answer

(a) Yes: $50 + 51 = 101$
Yes: $33 + 34 + 35 = 102$
Yes: $18 + 19 + 20 + 21 + 22 = 100$

(b) It is possible to make all numbers, except for powers of two, which are impossible.

Solution

(a) We can write 101 as $50 + 51$. This generalises to any odd number $2n + 1$, which can be written as $n + (n + 1)$, provided $n > 0$.

We can write 102 as $33 + 34 + 35$. This generalises to any multiple of three, as $3n$ can be written as $(n - 1) + n + (n + 1)$, provided $n > 1$.

We can write 100 as $18 + 19 + 20 + 21 + 22$. This generalises to any multiple of five, as $5n$ can be written as $(n - 2) + (n - 1) + n + (n + 1) + (n + 2)$, provided $n > 2$.

> These examples suggest some patterns that will allow us to write many numbers in the required form. In particular, the examples of 100 and 102 suggest that having an odd factor can give a way to write a number as a sum of consecutive integers.

(b) We can consider what happens when we sum together k consecutive numbers. There are two cases to consider, depending on whether k is even or odd.

- If k is odd, then we can write $k = 2m + 1$. Then, consider the middle number a of this set of k numbers. Since the first number is at least 1, we need $a > m$ for this to occur. Then, the numbers are $a - m, a - m + 1, \dots, a, \dots, a + m$. The average of these numbers is a and there are $2m + 1$ of these, so the sum is $a(2m + 1)$.
- If k is even, then we can write $k = 2m$. Now, suppose the middle pair of numbers are a and $a + 1$. Then, the numbers are $a - m + 1, a - m + 2, \dots, a, a + 1, \dots, a + m$, which is valid provided $a \geq m$. The average of these numbers is $a + \frac{1}{2}$, and there are $2m$ of them, so the sum is $2m\left(a + \frac{1}{2}\right) = m(2a + 1)$.

Both of these formulations are the product of an odd number greater than 1 with some other number. This means all that can be written in the required form have an odd factor greater than 1. Furthermore, if $n = p(2q + 1)$ with either $p > q$ or $p \leq q$, then we can write n as the sum of consecutive numbers, so every number with an odd factor greater than 1 can be written in this way.

All that remains is to compute which numbers have no odd factor apart from 1. This is the same as having no odd prime factors, which is equivalent to being a power of 2. Hence we can express all positive integers as the sum of consecutive integers apart from powers of 2, which are impossible.

United Kingdom
Mathematics Trust

Mentoring Scheme

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Mary Cartwright

Sheet 3

Solutions and comments

This programme of the Mentoring Scheme is named after Dame Mary Lucy Cartwright (1900–1998).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Cartwright.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

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University of Leeds, Leeds LS2 9JT

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1. (a) Explain why all the following statements are equivalent:

i. a and b have the same remainder when divided by n;

ii. $a - b$ is divisible by n;

iii. $a = b + tn$ for some whole number t.

Can you find any other ways of expressing this concept?

This property can be expressed by saying that a and b are *congruent modulo* n. This is written as $a \equiv b \pmod{n}$.

(b) Can you find a small integer x such that $x \equiv 742388 \pmod{9}$?

(c) i. Show that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$.

ii. What is the remainder when $426379 + 2177354$ is divided by 7? Try to find it without actually performing the addition!

(d) i. Show that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then $a \times c \equiv b \times d \pmod{n}$.

ii. What is the remainder when 426379×2177354 is divided by 7? Try to do this without actually performing the addition!

ANSWER

(b) 5

(c) ii. 6

(d) ii. 1

SOLUTION

(a) If a and b have the same remainder r when divided by n, then $a \div n = x$ remainder r and $b \div n = y$ remainder r.

This can be most clearly expressed as: $a = nx + r$ and $b = ny + r$.

Subtracting the two equations, we obtain $a - b = nx - ny = n(x - y)$, so $a - b$ is divisible by n.

We may rearrange the above to give $a = b + n(x - y) = b + nt$, where $t = x - y$.

If $a = b + nt$, then $a \div n = (b + nt) \div n = t + (b \div n)$, so a and b have the same remainder when divided by n.

This means that from any of these statements, we can reach any of the others.

(b) $742388 \div 9 = 82487$ remainder 5. This means that 5 and 742388 have the same remainder, 5, when divided by 9, so we have $5 \equiv 742388 \pmod{9}$.

(c) i. We know that $a \equiv b \pmod{n}$, which means that $a = b + tn$. Likewise, $c \equiv d \pmod{n}$, so $c = d + un$. This means $a + c = b + d + tn + un = b + d + (t + u)n$. But now we can use the definition of congruence modulo n to say that $a + b \equiv b + d \pmod{n}$.

ii. By dividing the numbers by 7, we can determine that $426379 = 60911 \times 7 + 2 \equiv 2 \pmod{7}$ and $2177354 = 311050 \times 7 + 4 \equiv 4 \pmod{7}$. Then we can use the result we obtained in the previous part to say that $426379 + 2177354 \equiv 2 + 4 \equiv 6 \pmod{7}$. Thus the remainder when divided by 7 is 6.

(d) i. We know that $a \equiv b \pmod{n}$, which means that $a = b + tn$. Likewise, $c \equiv d \pmod{n}$, so $c = d + un$. This means $ac = (b + tn)(d + un) = bd + bun + ctn + tun^2 = bd + (bu + ct + tun)n$. But now we can use the definition of congruence modulo n to say that $ac \equiv bd \pmod{n}$.

ii. We can use the result we obtained in the previous part to say that $426379 \times 2177354 \equiv 2 \times 4 \equiv 8 \equiv 1 \pmod 7$, so its remainder when divided by 7 is 1.

2. (a) A sequence t_n is formed using the following rules:

- $t_0 = 2$;
- $t_{n+1} = \frac{1}{t_n}$.

What is the value of t_{1001}?

(b) A different sequence u_n is formed using the following rules:

- $u_0 = 2$;
- for n even, $u_{n+1} = \frac{1}{u_n}$;
- for n odd, $u_{n+1} = 1 - u_n$.

What is the value of u_{1001}?

ANSWER

(a) $\frac{1}{2}$

(b) -1

SOLUTION

(a)

The first thing worth trying in this sort of question is to see if there is a recognisable pattern. If we do this, we obtain:

$$t_0 = 2, t_1 = \frac{1}{2}, t_2 = \frac{1}{\frac{1}{2}} = 2, t_3 = \frac{1}{2}, t_4 = 2, t_5 = \frac{1}{2}.$$

The sequence appears to alternate between 2 and $\frac{1}{2}$. However, to know that this always happens, we need a proof.

One thing to observe is that t_{n+1} depends only on t_n. Then, since we have $t_1 = \frac{1}{2}$ and $t_2 = 2 = t_0$, the start of the sequence has repeated. We know that this will keep on happening, as each term depends only on the previous one.

Alternatively, you could observe that $t_{n+2} = \frac{1}{t_{n+1}} = \frac{1}{\frac{1}{t_n}} = t_n$.

In either case, $t_{1000} = t_{998} = \ldots = t_2 = t_0 = 2$, so $t_{1001} = \frac{1}{2}$.

(b) Each term u_{n+1} depends only on the previous one, namely u_n, and the parity of n (whether n is odd or even). This allows us to construct the following table:

n	u_n	Parity of n	u_{n+1}
0	2	Even	$\frac{1}{2}$
1	$\frac{1}{2}$	Odd	$\frac{1}{2}$
2	$\frac{1}{2}$	Even	2
3	2	Odd	-1
4	-1	Even	-1
5	-1	Odd	2
6	2	Even	$\frac{1}{2}$

Then in the final row we have reached the same value of u_n and the same parity of n. This means that this sequence will repeat every six terms. Then, as $1001 = 6 \times 166 + 5$, $u_{1001} = u_5 = -1$.

3. A daughter and her father both have ages that consist of two digits. The daughter writes down her father's age followed by her own age, producing a four digit number. When she subtracts from this the difference of their ages, she obtains the answer 4289. How old are the daughter and her father?

Answer 16 and 43

Solution

Write d for the daughter's age and f for the father's age. Then, writing these next to each other is the same as writing $100f + d$. As the father is older than the daughter, the difference between their ages is $f - d$. Subtracting this from the four-digit number gives

$$(100f + d) - (f - d) = 4289.$$

Collecting like terms yields

$$99f + 2d = 4289.$$

Now, as the daughter's age is a two-digit number, she is at least 10 and at most 99, that is $10 \le d \le 99$. This means

$$4289 - 2 \times 99 \le 4289 - 2d = 99f \le 4289 - 2 \times 10.$$

Consequently $4091 \le 99f \le 4269$, so $41\frac{32}{99} \le f \le 43\frac{12}{99}$. Therefore $f = 42$ or $f = 43$.

If $f = 42$, then $2d = 4289 - 99f = 4289 - 99 \times 42 = 4289 - 4158 = 131$. But this is not even and we know that d is a whole number. If $f = 43$, then $2d = 4289 - 99f = 4289 - 99 \times 43 = 4289 - 4257 = 32$, implying $d = 16$.

You can check that this does indeed give a solution. Writing the ages consecutively gives 4316. The difference in the ages is $43 - 16 = 27$. Then, $4316 - 27 = 4289$ as required.

Alternative

A different way of solving part of this question is to make use of modular arithmetic, which we met in Question 1. We know that

$$2d + 99f = 4289$$

Then, working modulo 99, we have that $99 \equiv 0 \pmod{99}$ and $4289 \equiv 32 \pmod{99}$, so

$$2d + 0f \equiv 32 \pmod{99}$$

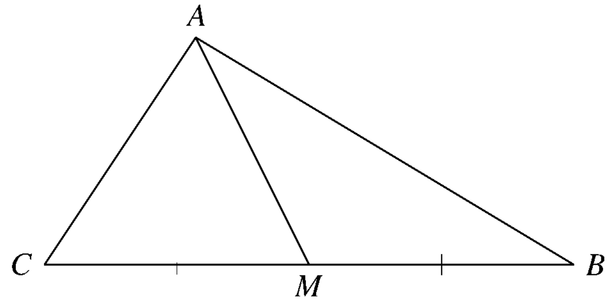
We cannot divide when working with congruences, but we can multiply, so let us multiply both sides by 50. This gives $100d \equiv 1600 \pmod{99}$. Reducing these terms gives $d \equiv 16 \pmod{99}$. Then $d = 16 + 99t$, but d has two digits, so $d = 16$. Then, we can calculate the value of f to be 43 and can check in the same way as before.

> Since $2 \times 50 \equiv 1 \pmod{99}$, we say that 50 is the *multiplicative inverse* of 2 modulo 99. Multiplicative inverses are important in solving modular equations such as $2d \equiv 16 \pmod{99}$. They can also be used to ensure there is only one solution.

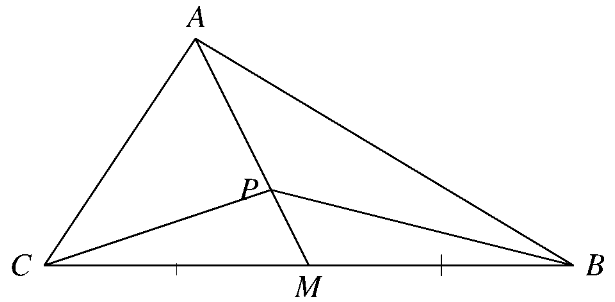
4. *The notation* $[ABC]$ *denotes the area of the triangle* ABC.

In the triangle ABC, M is the midpoint of BC.

(a) Prove that $[ABM] = [AMC]$.

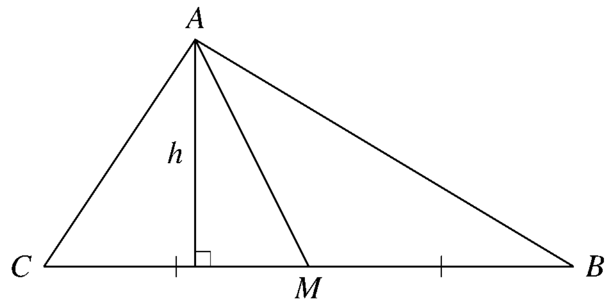


(b) Suppose that P lies on the line segment AM. Prove that $[ABP] = [APC]$.



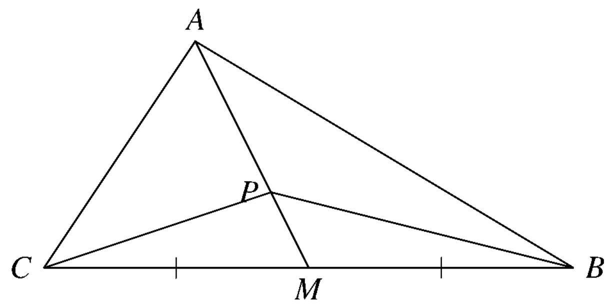
Solution

(a) We can draw in the perpendicular height from A onto the line BC.



We know that this is the same height for both the triangles ABM and AMC; let us call this height h. Likewise, as M is the midpoint of BC, we know that $CM = MB$. This means that $\frac{1}{2}hCM = \frac{1}{2}hMB$ and so $[ABM] = [AMC]$, i.e. the triangles ABM and AMC have the same area.

(b)



We can apply the first part of the question to tell us that $[ABM] = [AMC]$. We can then apply the

first part of the question to triangle PBC to tell us that $[PBM] = [PMC]$ also. This means we can subtract this pair of areas to tell us about the areas we require:

$$[APC] = [AMC] - [PMC] = [ABM] - [PBM] = [ABP].$$

This is what we wanted, so we are done!

5. Can you find a four digit perfect square such that the first two digits are the same and the last two digits are also the same? Are there any others?

ANSWER 7744 is the only solution

SOLUTION

Informally, we may write a four-digit number, where the first two digits are the same and the last two digits are the same, as $AABB$. We can then see that this may be factorised as $11 \times A0B$.

Algebraically, '$AABB$' can be expressed as $1000A + 100A + 10B + B = 1100A + 11B = 11 \times (100A + B)$, where we have taken out the factor of 11 informally expressed above.

Since this is a square number, it must have another factor of 11, so $100A + B$ must be divisible by 11. But $100A + B = 11(9A) + (A + B)$, so $A + B$ must be divisible by 11.

Since A and B are both digits and we have a four-digit number, we have the inequalities: $1 \leq A \leq 9$ and $0 \leq B \leq 9$.

These inequalities imply that the sum $A + B$ must therefore be equal to 11, which we can express as $B = 11 - A$.

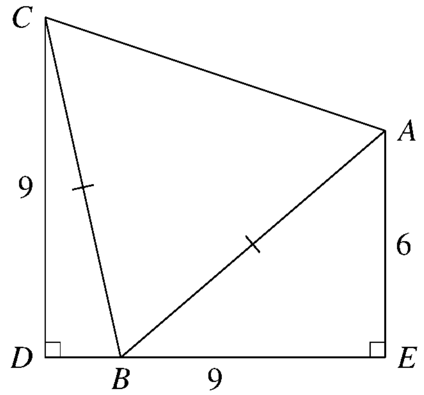
We then substitute for B, to obtain $100A + B = 99A + 11 = 11(9A + 1)$.

This means the original square number is $11^2 \times (9A + 1)$. It follows that $9A + 1$ must also be a square number. Since A is a digit, we can work out all eight possibilities. (Note that A is not 1 as then B would be 10.) The eight possibilities are listed in the following table.

A	$9A + 1$	Square?
2	19	No
3	28	No
4	37	No
5	46	No
6	55	No
7	64	Yes
8	73	No
9	82	No

This means that the only possible solution is $A = 7$, with $B = 11 - A = 4$. Thus the original number must be $7744 = 88^2$.

6. In the diagram below, isosceles triangle ABC has had right angled triangles AEB and BDC constructed on its sides, such that DBE is a straight line.

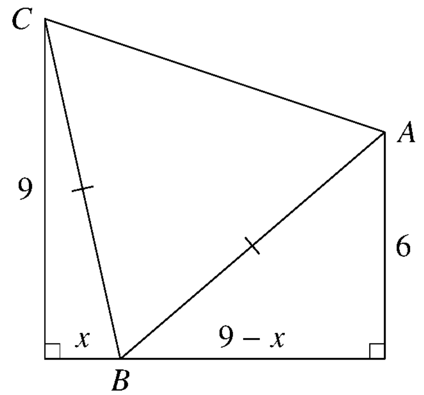


If $CD = 9$, $DE = 9$ and $EA = 6$, then what is the length of AB?

Answer $\sqrt{85}$

Solution

We can start by labelling the length DB as x. This means that BE has length $9 - x$.



Now, we can apply Pythagoras' theorem in triangles CBD and AEB to give:

$$AB^2 = AE^2 + BE^2 = 6^2 + (9 - x)^2 = x^2 - 18x + 117;$$
$$BC^2 = BD^2 + CD^2 = x^2 + 9^2 = x^2 + 81.$$

We also know that ABC is an isosceles triangle with $AB = BC$. Hence

$$x^2 - 18x + 117 = x^2 + 81.$$

Rearranging this equation gives

$$18x = 36$$

and so $x = 2$. Then, using the formula above, $AB = \sqrt{6^2 + (9 - x)^2} = \sqrt{6^2 + 7^2} = \sqrt{36 + 49} = \sqrt{85}$.

7. A postwoman has seven letters to deliver to each of the seven dwarves. Unfortunately, she is rather inaccurate in her deliveries.

(a) How many ways are there in which she can deliver exactly two of the letters incorrectly?

(b) How many ways are there in which she can deliver exactly three of the letters incorrectly?

Answer

(a) 21

(b) 70

Solution

In this question we will use some notation that we met last month. We will write $\binom{n}{k}$ (n choose k) for the number of ways of selecting k items from a set of n items. Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

(a) There are $\binom{7}{2} = 21$ ways to choose the two dwarves that will receive incorrect letters: the other five must all receive the letters addressed to them.

Then for these two dwarves, neither has received their own letter, so each must have received that sent to the other.

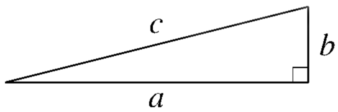
Therefore, there are $21 \times 1 = 21$ ways that the postwoman could have delivered the letters.

(b) There are $\binom{7}{3} = 35$ ways to choose the three dwarves that received incorrect letters.

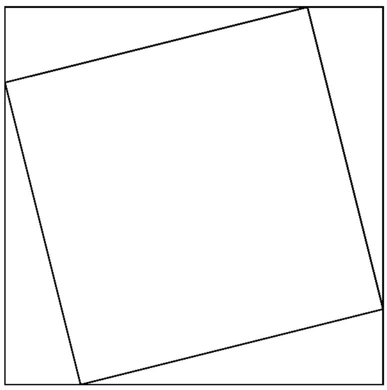
For these three dwarves (let us call them Grumpy, Happy and Sneezy) who receive the wrong letters, we need to count how many ways this can happen. If Grumpy's letter is sent to Happy, then Sneezy must receive Happy's letter, as he does not receive his own. This leaves Grumpy to receive Sneezy's letter. Otherwise, Grumpy's letter must be sent to Sneezy, which in the same way gives one solution. This means there are two ways for the three dwarves to receive the wrong letters.

Therefore there are $35 \times 2 = 70$ ways for three dwarves to receive the wrong letters.

8. Consider the following right angled triangle:



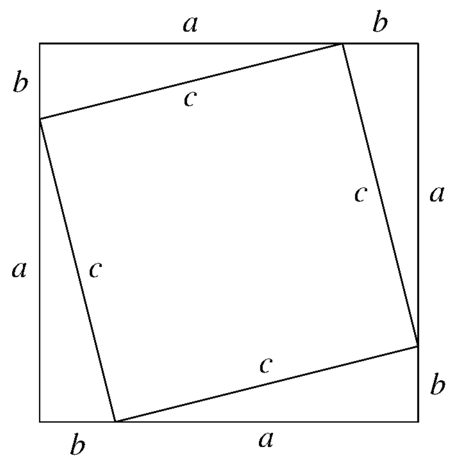
Four of these triangles can be placed together in the following square arrangement:



Can you use this diagram to prove Pythagoras' theorem; that is to prove that $a^2 + b^2 = c^2$?

Solution

We can label the lengths on the diagram as follows.



We now can calculate the area of the whole square in two different ways:

- The square has side-length $a + b$, so it has area $(a + b)^2 = a^2 + 2ab + b^2$.
- The square consists of four small triangles and the central square of side-length c. The small square has area c^2. Each of the triangles has base b and height a, so area $\frac{1}{2}ab$. This means that the total area is $c^2 + 4 \times \frac{1}{2}ab = c^2 + 2ab$.

Since the two expressions are both for the same area, they must be equal. Therefore $a^2+2ab+b^2 = c^2+2ab$. Subtracting $2ab$ from both sides yields $a^2 + b^2 = c^2$, which is the theorem of Pythagoras that we wanted to prove.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Mary Cartwright

Sheet 4

Solutions and comments

This programme of the Mentoring Scheme is named after Dame Mary Lucy Cartwright (1900–1998).

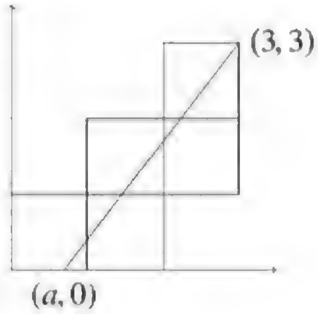
See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Cartwright.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

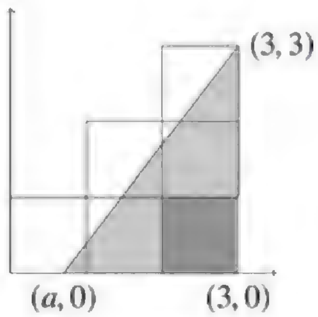
☎ 0113 343 2339 0113 343 5500 mentoring@ukmt.org.uk www.ukmt.org.uk

1. Five unit squares are arranged as shown in the diagram below. A line is drawn from the point $(3, 3)$ to the point $(a, 0)$ which divides the area of the five squares into two equal areas. What is the value of a?



Answer $\frac{2}{3}$

Solution



There are five squares, each of area 1, so their total area must be 5. This means that we want the lighter shaded area to have area $\frac{5}{2}$.

The darker shaded area is another unit square, so again has area 1. This means that we want the total shaded area to be $\frac{5}{2} + 1 = \frac{7}{2}$.

However, we can calculate this as the area of the shaded triangle, which has a height of 3 and a base of length $3 - a$. This means it has area $\frac{1}{2} \times 3(3 - a)$. This area must be equal to $\frac{7}{2}$, so

$$\frac{3}{2}(3 - a) = \frac{7}{2}$$

Multiplying both sides of this equation by $\frac{2}{3}$ gives

$$3 - a = \frac{7}{3}$$

Therefore $a = \frac{2}{3}$, by subtracting each side from 3.

2. Before starting to paint, Rhiannon had 130 grams of blue paint, 164 grams of red paint and 188 grams of white paint. She painted four equally sized stripes on the wall, making a red stripe, a blue stripe, a white stripe and a pink stripe. To make the pink stripe she used only red and white paint, but not necessarily in equal amounts.

If Rhiannon ended up with equal amounts of red, white and blue paint, what is the total amount of paint she had left.

Answer 114 grams

SOLUTION

> The key first step in this question is to work out how to convert the words in the question into some algebra, and this means working out what the key things are that we don't know. Firstly, there is the amount of each type of paint that Rhiannon ends up with. We also don't know how much paint is used to make each stripe. A third unknown is the amounts of red and white paint used to make the stripe of pink paint.

Let us use the following variable names:

- p for the amount of paint of each colour left at the end.
- s for the amount of paint used in each stripe.
- r for the amount of red paint used in the pink stripe.

Then, we can say the following statements about the different coloured paints:

- There initially was 130 grams of blue paint. When s grams were used, this left p grams.
- There initially was 164 grams of red paint. When $s + r$ grams were used, this left p grams.
- Since there were s grams of paint used in the pink stripe, r of which were red, the other $s - r$ were white. This means that, of the initial 188 grams of white paint, $2s - r$ were used, leaving p grams.

This gives the following equations:

$$\begin{aligned} p + s &= 130 \\ p + s + r &= 164 \\ p + 2s - r &= 188 \end{aligned}$$

Then, subtracting the first equation from the second tells us that $r = 34$. Substituting this into the third equation tells us that $p + 2s = 222$. Subtracting the first equation away from this gives $s = 92$. Then, substituting into the first equation gives that $p = 38$.

We can check that this is a valid solution. Each stripe used 92 grams of paint, so the one blue stripe left $130 - 92 = 38$ grams of blue paint. The red stripe used 92 grams, and another 34 towards the pink stripe, leaving $164 - 92 - 34 = 38$ grams of red paint. There was $92 - 34 = 58$ grams of white paint in the pink, so there was $188 - 92 - 58 = 38$ grams of white paint remaining.

This means there was a total of $3 \times 38 = 114$ grams of paint remaining.

3. (a) How many four digit numbers are there with all digits different and where the first and last digits have a difference of exactly 2?

(b) What about if the middle digits also have a difference of exactly 2 (in addition to all the digits being different and the first and last still having a difference of 2)?

ANSWER

(a) 840
(b) 164

SOLUTION

(a) If the first and last digits must differ by exactly 2, then the smaller of them can be any number between 0 and 7, with the larger one two bigger. This gives a total of eight choices.

Since these can go in either order, there are a total of $8 \times 2 = 16$ ways of ordering them. However, numbers of the form $0 * *2$ are not valid as they start with a 0, so that leaves 15 different ways.

For each of these, the two digits in the middle must be chosen to be different from the end ones and from each other. There are eight other digits available for the first of these, and then seven remain for the second. This means the total number of arrangements is $15 \times 8 \times 7 = 840$.

(b) The available pairs which differ by exactly two are:

$$02, 13, 24, 35, 46, 57, 68, 79$$

We need to choose two of these pairs of digits, but these need to share no common digits. There are $\binom{8}{2} = 28$ ways of choosing the two different digits. However, the following pairs of pairs have a common digit:

02	and	24
13	and	35
24	and	46
35	and	57
46	and	68
57	and	79

This means there are $28 - 6 = 22$ ways of choosing the two pairs.

Then, we have to choose one of these pairs to go first, and can choose the order of each of the pairs, each of which can be done in two ways. This gives us a total of $22 \times 2^3 = 176$ different options.

However, we haven't excluded those cases where this number would have a leading zero. This would mean that the final digit must be a 2, so we can choose any of the pairs 13, 35, 46, 57, 68, and 79 for the middle digits. Each of these can be placed in either way round, so there are a total of $6 \times 2 = 12$ options.

This means the overall number of possibilities is $176 - 12 = 164$.

4. What is the remainder when 4444^{4444} is divided by 9?

ANSWER 7

SOLUTION

> Last month you'll have met the concept of modular arithmetic. We'll make use of this in order to make this rather scary calculation a whole lot easier.

We can work modulo nine: as $4444 = 493 \times 9 + 7$, we know that $4444 \equiv 7 \pmod 9$.

Then, since we can think of 4444^{4444} as 4444 multiplied together 4444 times, this means we can write:

$$4444^{4444} \equiv 7^{4444} \pmod 9$$

Now, calculating the first few powers of seven modulo 9, we can see that:

$$\begin{aligned} 7^1 &\equiv & 7 \pmod 9 \\ 7^2 &\equiv 7\times 7 \equiv 49 \equiv & 4 \pmod 9 \\ 7^3 &\equiv 7\times 4 \equiv 28 \equiv & 1 \pmod 9 \end{aligned}$$

Then, as $4444 = 3 \times 1481 + 1$, we can write:

$$4444^{4444} \equiv 7^{4444} \equiv \left(7^3\right)^{1481} \times 7 \equiv 1^{1481} \times 7 \equiv 7 \pmod 9$$

This means that the remainder when 4444^{4444} is divided by 9 is 7.

5. (a) Suppose that the integer n is both a square number and a cube number.

i. If p is a prime dividing n, what can you say about the power of p in the prime factorisation n?

ii. Why must n be the sixth power of some integer?

(b) Suppose that a and b are positive integers greater than 1 such that $\sqrt{a\sqrt{a\sqrt{a}}} = b$. What is the smallest possible value of $a + b$?

Answer

(a) i. It must be a multiple of 6

(b) 384

Solution

(a) i. Since n is a square number, we can write $n = x^2$. Then, in the prime factorisation of x, p is raised to some power, say a, so $x = p^a r$, where r is not divisible by p. Then, we can write $n = x^2 = p^{2a}r^2$.

Since n is a cube number, we can write $n = y^3$. Then, in the prime factorisation of y, p is raised to some power, say b, so $y = p^b s$, where s is not divisible by p. Then, we can write $n = y^3 = p^{3b}s^3$.

This means that we know that the power of p must be both even and a multiple of three, from the two expressions that we have calculated. This means the power of p must be a multiple of six.

ii. Suppose n has the prime factorisation $p_1^{a_1}p_2^{a_2}\ldots p_k^{a_k}$. Then, we can apply the above logic to each of the primes to tell us that $n = p_1^{6b_1}p_2^{6b_2}\ldots p_k^{6b_k} = \left(p_1^{b_1}p_2^{b_2}\ldots p_k^{b_k}\right)^6$, with each b_i an integer. This tells us that n is indeed a sixth power.

(b)

> It may not be immediately apparent what the best way to start tackling a question such as this is. The square roots in the equation look somewhat scary, so maybe the best idea is to square both sides, to try and remove these:

Squaring both sides of the equation gives us

$$a\sqrt{a\sqrt{a}} = b^2$$

Then, we can square the equation again, to give

$$a^3\sqrt{a} = b^4$$

Squaring for a third time gives

$$a^7 = b^8$$

Now, notice that as a increases in this equation, so does b (remembering that we know that both are positive), so the smallest value of $a + b$ will be obtained for the smallest value of a.

We know that a^7 must be an eighth power of some number, as well as a seventh power. We can work exactly as in the first part of the question, to tell us that a^7 must be a 56$^{\text{th}}$ power. Since $a > 1$, $a^7 > 1$, so the smallest possible value of this is 2^{56}.

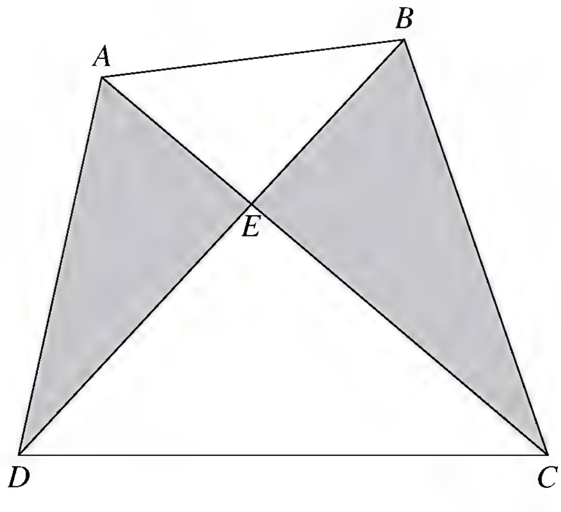
In this case, $a = 2^8 = 256$ and $b = 2^7 = 128$. This gives the minimal value of $a+b$ as $256+128 = 384$.

6. Suppose $ABCD$ is a convex quadrilateral (no internal angle is more than 180°). If E is the point of intersection of the diagonals AC and BD, the triangles AED and BEC have equal areas. Prove that $\angle ABD$ and $\angle BDC$ are equal

SOLUTION

Putting the information from the question onto a diagram gives the following, with the shaded areas equal:

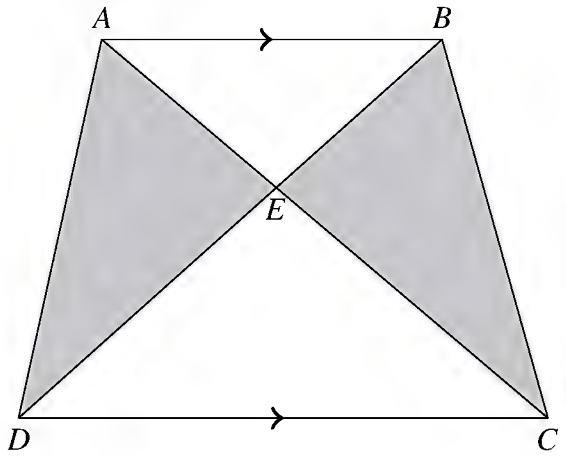
> In this type of problem, it's always a good idea to draw yourself a good, clear diagram so that you can see what the problem is asking. The diagram should be large and clear enough to make sure none of the important details are lost.



> Currently, we know that the two shaded triangles have the same area, but we need to convert this to a statement about lengths. If we could use these to find two triangles with the same area which also shared a base, then we could deduce that these two triangles have the same height.

We know that triangles AED and BCE have the same area. We can attach the triangle DEC to both, which tells us that triangles ACD and BCD have the same area.

ACD and BCD both have the same base also, namely CD. This means that they must have equal heights, so A and B have the same distance above CD. This means AB is parallel to CD.



Now, we can use these parallel lines to tell us something about the angles in the diagram. In particular, we know that the angles $\angle ABD$ and $\angle BDC$ are alternate angles, so they must be equal.

> **7.** A marching band is having difficulty lining up for a parade. When they line up in rows of 3, one person is left over. When they line up in rows of 4, two people are left over. When they line up in rows of 5, three people are left over. When they line up in rows of 6, four people are left over. However, the band is able to line up in rows of seven with nobody left over. What is the smallest number of marchers in the band?

ANSWER 238

SOLUTION

Let's start with the first two pieces of information, when the band lines up in rows of threes and fours. Let us suppose that there are n people in the band.

- When the band line up in rows of three, there is one person left over. This tells us that n is one more than a multiple of 3, or $n \equiv 1 \pmod 3$.
- When the band line up in rows of four, there are two people left over. This tells us that n is two more than a multiple of 4, or $n \equiv 2 \pmod 4$.

Now, we can look for a number that has both of these properties. Since the number must be two more than a multiple of 4, we can try 2 and 6, which are not congruent to 1 modulo 3, before arriving at 10, which is.

Suppose the true solution was different to this, so there must be a different number of people. These extra people must form more complete rows in both the cases where the marchers line up in rows of three and in rows of four. This means precisely that there must be a multiple of 12 extra people, so we know that $n \equiv 10 \pmod{12}$.

> This is an example of the *Chinese Remainder Theorem*, which says that if $n \equiv a \pmod r$ and $n \equiv b \pmod s$ with r and s having no common factors, then there is a unique value $c \pmod{rs}$ such that $n \equiv c \pmod{rs}$.
>
> This is a very useful result for questions of this type. You might like to have a think about how you would go about proving it - it is very tricky though!

Then, lining the band up in rows of sixes, each set of 12 forms two rows of six, and the remaining ten form a row of 6 with four left over. This means we must already satisfy the criterion with rows of six.

For rows of five, we know that $n \equiv 3 \pmod 5$ and $n \equiv 10 \pmod{12}$. One possible solution would be 58, which you can find by working though all the numbers congruent to $10 \pmod{12}$ to find one that is also congruent to $3 \pmod 5$. Also, for any solution the extra rows must form a multiple of both 12 and 5, so must be a multiple of 60. This tells us exactly that $n \equiv 58 \pmod{60}$.

We now need the smallest number congruent to $58 \pmod{60}$ which is also a multiple of 7. The first few numbers congruent to $58 \pmod{60}$ are: 58, 118, 178 and 238. The first three of these are not divisible by 7, but 238 is. Therefore this is the smallest solution.

We can easily check that 238 satisfies all the conditions from the question, so it is indeed the smallest solution.

Alternative

If there were two more people in the band, then it would be able to line up in rows of 3, 4, 5 or 6, so $n + 2$ must be divisible by the lowest common multiple of these numbers, which is 60.

Then, we need n to be divisible by 7 also, so we need to find the smallest multiple of 60 that is two more than a multiple of 7. None of $60 \times 1 - 2 = 58$, $60 \times 2 - 2 = 118$ and $60 \times 3 - 2 = 178$ are multiples of 7, but $60 \times 4 - 2 = 238 = 7 \times 34$ is a valid solution.

8. Each square in a 9×11 grid is to be coloured either white or black in such a way that every 2×3 rectangle contains exactly two black squares, and every 3×2 rectangle contains exactly two black squares. How many different arrangements of tiles are there?

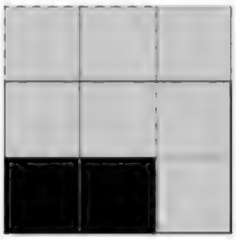
Answer 6

Solution

> This is one of those questions where there's no obvious way to start it, so the best thing to do might be to try and draw some examples that work, and see what you notice. One thing that you might find is that the black squares seem to end up diagonally adjacent, but not in a straight line either next door or just with one white tile in between.
>
> So maybe we can work out why each of these things always happen, which will tell us what is occurring.

Two adjacent black squares

Let us start by considering what happens if we have two tiles that are adjacent to each other which are black. One of the 3×3 squares with these in the top-right, top-left, bottom-right or bottom-left must lie entirely in the grid. By rotation and reflection we can get this section to look like:

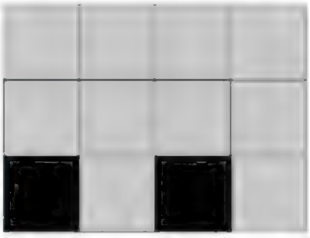


Then, we can look at the lower and left hand rectangles marked on the diagram. Since each of these already contains two black squares, the others must all be white. This gives the following diagram:

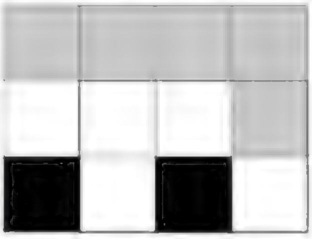
Then consider the top rectangle, marked on the diagram. This can contain at most 1 black square, which is not allowed by the statement of the question, so the initial assumption, that we had two adjacent squares black, is not allowed.

Two black squares separated by 1

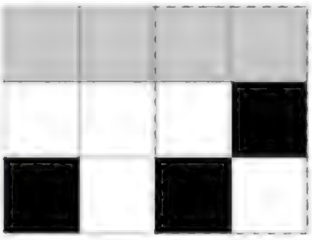
Suppose instead that we have two squares that are separated by one which are both black. We can place these into a 4×3 rectangle using rotation and reflection, as in the diagram below:



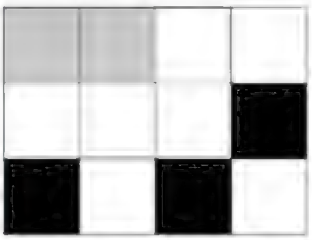
Then, we know from above that none of the adjacent squares to these can be black, so must be white. The marked rectangle contains two black squares, so all the others are white. This gives:



Then, the last square in the solid rectangle must be black. This then produces the following diagram:



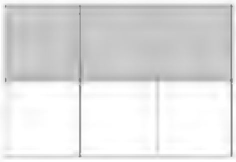
Now, the dashed rectangle contains two black squares, so the other two squares in it must both be white.



The final two squares in the solid rectangle must be black, as all the other four squares are white. But, this would produce two adjacent squares which are both black, and we know from above that this is impossible. This means we cannot have two squares at a distance of two from each other, both of which are black.

Three consecutive white squares

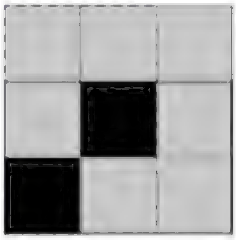
Suppose there were three consecutive white squares in a row. These lie inside some 2×3 or 3×2 grid - by rotation and reflection the diagram below can be obtained:



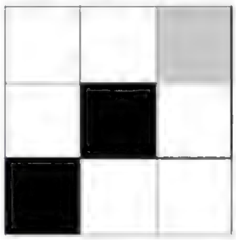
Then, this must have two squares in the row above coloured black. But these cannot be adjacent or have a gap of one between them, so two cannot be chosen. This means we cannot have three white squares in a row.

Black Diagonals Continue

Suppose there are two black squares in a diagonal, then we can consider the next square in that diagonal.



Each of the marked rectangles contains two black squares, so the others must be white. This gives:



The last square then must be black, from the marked rectangle. This tells us that this diagonal continues.

Every third square is black

From a black square, the next two squares in a row must both be white, as they cannot be black. But then, we cannot have three squares white in a row, so the next square must be black.

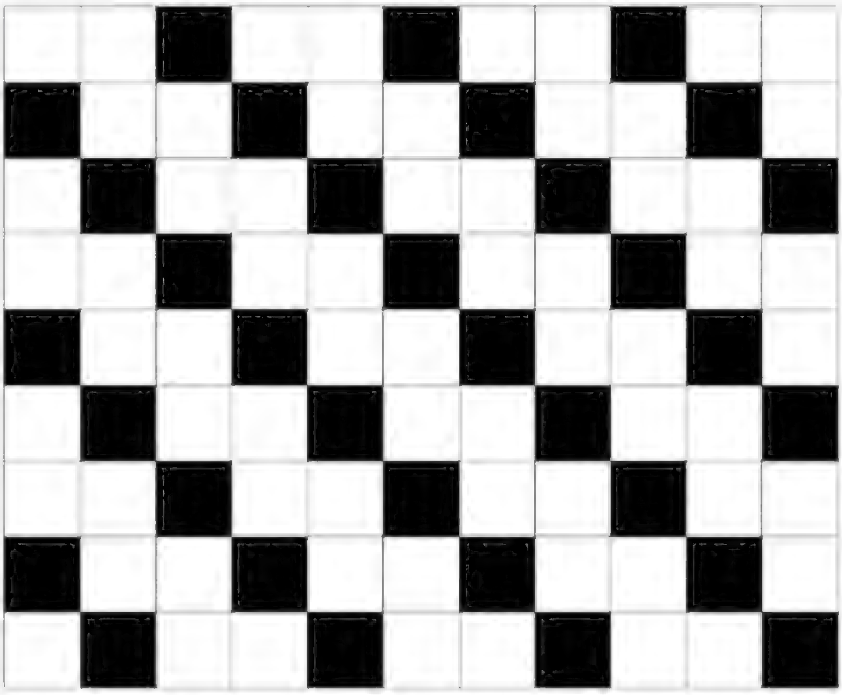
Counting Combinations

Now we've established a number of rules that the pattern must follow, we can then start to count combinations.

Both the 3×1 rectangles marked must contain exactly one black square, and these cannot be in line - there are six ways of achieving this.

But then, from each of these we can expand out to fill the entire grid - we can continue the diagonal lines that are obtained, and make every third square in the rows black. In this way we end up with every third diagonal consisting of black squares, and all the others white. This means there are exactly six choices - depending on which direction the diagonals go in (up and left or up and right) and three choices of diagonal.

An example of this is:



United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Mary Cartwright

Sheet 5

Solutions and comments

This programme of the Mentoring Scheme is named after Dame Mary Lucy Cartwright (1900–1998).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Cartwright.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

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1. (a) Prove that an integer is divisible by 9 exactly when the sum of its digits is divisible by 9.

(b) Prove that an integer is divisible by 11 exactly when the sum of the digits in the even positions minus the sum of the digits in the odd positions is divisible by 11.

Solution

(a) Write the number, in digit form, as $A_nA_{n-1}\dots A_1A_0$. Then, we can express this algebraically as a sum of multiples of powers of 10:

$$10^nA_n + 10^{n-1}A_{n-1} + \dots 10A_1 + A_n.$$

Now, we need to consider the value of this expression modulo 9, using the fact that $10 \equiv 1 \pmod 9$. We have

$$\begin{aligned}10^nA_n + 10^{n-1}A_{n-1} + \dots 10A_1 + A_n &\equiv 1^nA_n + 1^{n-1}A_{n-1} + \dots + 1A_1 + A_0 \pmod 9\\ &\equiv A_n + A_{n-1} + \dots + A_1 + A_0 \pmod 9.\end{aligned}$$

This is equivalent to saying that, on division by 9, the sum of the digits has the same remainder as the initial number. In particular, the sum of the digits is divisible by 9 exactly when the initial number is.

(b) We can use the same technique working modulo 11, this time using the fact that $10 \equiv -1 \pmod{11}$. We have

$$10^nA_n + 10^{n-1}A_{n-1} + \dots 10A_1 + A_n \equiv (-1)^nA_n + (-1)^{n-1}A_{n-1} + \dots + (-1)A_1 + A_0 \pmod{11}.$$

For this to be congruent to 0, we need the total of the positive and the negative terms to be the same modulo 11. This is exactly the same thing as saying that the sum of the digits in the odd positions minus the sum of the digits in the even positions must be divisible by 11. This is the required result.

2. (a) How many positive integers up to and including 1000 are divisible by either 3 or 5?

(b) How many positive integers up to and including 1000 are divisible by either 3, 5 or 7?

Answer

(a) 467

(b) 543

Solution

The key ingredient to solving this question is the (informal) idea of 'multiple counting'.

The approach for part (a) is to count all of the items, where some items may be counted twice, then proceed to subtract the number of items that have been counted twice. In the end we ensure that we count each of the items precisely once.

Another name for this technique is the *inclusion-exclusion principle,* which says the following:

If you have two sets A and B, then the number of items appearing in at least one of these sets is the sum of the set sizes minus the number of elements that we have counted twice. Those elements which have been counted twice are precisely those elements appearing in both sets.

Symbolically, we write this as

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

> Here $A \cup B$, read as A *union* B, is the set of all elements appearing in at least one of A and B. $A \cap B$, read as A *intersection* B, is the set of all elements in both A and B.
>
> For success in part (b) of this question, we need to generalise the multiple counting idea. We now have three collections and so some items can be counted once, twice or even three times. The generalisation is explained in the solution.

(a) Let A be the set of positive integers up to 1000 divisible by 3 and B the set of positive integers up to 1000 divisible by 5. Then what we are looking for is the set of numbers divisible by 3 or by 5, in other words $A \cup B$.

A multiple of 3 occurs every third number. This means we need the number of complete sets of three integers less than or equal to 1000. If $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, there must be $\lfloor \frac{1000}{3} \rfloor = \lfloor 333\frac{1}{3} \rfloor = 333$ such sets. We can write this as $|A| = 333$.

We can do the same for multiples of 5: there are $\lfloor \frac{1000}{5} \rfloor = 200$ of these. We can write this as $|B| = 200$.

Adding these together gives a total of 533, but we have counted twice any number that is both a multiple of 3 and a multiple of 5. This needs to be corrected.

Being a multiple of both 3 and 5 is exactly the same as being a multiple of 15. There are $\lfloor \frac{1000}{15} \rfloor = \lfloor 66\frac{2}{3} \rfloor = 66$ of these, so $|A \cap B| = 66$. The result that we need is achieved by subtracting 66 to get the overall total of $533 - 66 = 467$.

Symbolically, we have

$$|A \cup B| = |A| + |B| - |A \cap B| = 333 + 200 - 66 = 467.$$

(b) Consider A and B as in the previous part, and let C be the set of positive integers up to 1000 that are divisible by 7. Then we want to know the size of $|A \cup B \cup C|$. If we consider $|A| + |B| + |C|$, then we have counted all the elements, but have double-counted all those appearing in two of the sets and triple-counted those in all three.

This suggests subtracting $|A \cap B| + |A \cap C| + |B \cap C|$ from $|A| + |B| + |C|$. This corrects the double counting for those numbers in exactly two sets. However, those in all three sets have been removed from the total three times: they do not appear at all. This means we need to count them back in, giving us overall

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Now we count the number of elements in each of the subsets involved

- $|C| = \lfloor \frac{1000}{7} \rfloor = 142$.
- $|A \cap C| = \lfloor \frac{1000}{21} \rfloor = 47$.
- $|B \cap C| = \lfloor \frac{1000}{35} \rfloor = 28$.
- $|A \cap B \cap C| = \lfloor \frac{1000}{105} \rfloor = 9$.

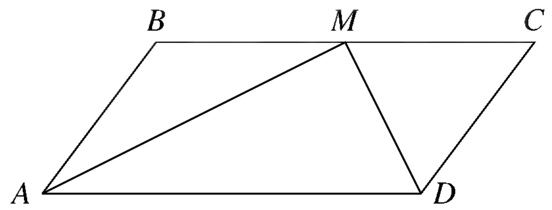
Combining these results,

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 333 + 200 + 142 - 66 - 47 - 28 + 9 \\ &= 543. \end{aligned}$$

3. In parallelogram $ABCD$, the bisector of $\angle A$ intersects the line segment BC at its midpoint M. Find, with justification, $\angle AMD$.

Answer 90°

Solution



Let us denote the angle $\angle BAM$ by θ.

> θ is the Greek letter *theta*, which is often used in mathematics, particularly to represent angles. You might also meet other Greek letters such as α (alpha) or β (beta). You have already met π though!

Since AM is the angle bisector of $\angle BAD$, $\angle MAD = \angle BAM = \theta$. Now, since the lines BC and AD are parallel, angles $\angle BMA$ and $\angle MAD$ are *alternate* and therefore equal. This implies that $\angle BMA = \angle MAD = \theta$.

It follows that $\angle BMA = \angle BAM$, so triangle ABM is isosceles, with $AB = BM$. Since M is the midpoint of BC, we know that $BM = MC$. We also know that opposite sides of a parallelogram have equal lengths, so $AB = DC$. Combining these gives $MC = DC$.

Now consider triangle MCD. Since $MC = DC$, it is isosceles and $\angle DMC = \angle CDM$. Since opposite angles in a parallelogram are equal, it follows that $\angle MCD = \angle DAB = 2\theta$. Then, as the other two angles are equal and angles in a triangle add to 180°, $\angle DMC = \frac{1}{2}(180° - \angle MCD) = \frac{1}{2}(180° - 2\theta) = 90° - \theta$.

Finally, we can use angles on a straight line at M (adding to 180°) to tell us that $\angle AMD = 180° - \angle BMA - \angle DMC = 180° - \theta - (90° - \theta) = 90°$.

4. In the last test but one of the year, Ellie scored 98 and her average score increased by 1. In the last test she scored 70, so that her average score decreased by 2. How many tests did she take during the year?

Answer 10

Solution

Suppose that Ellie had sat n tests before her final two, and at that stage had an average score of s. Since her average score is equal to the total score divided by the number of tests, her total score was ns.

After sitting one more test, her average score increased by 1 to $s + 1$, and she had sat $n + 1$ tests. Her total score must therefore have been $(n + 1)(s + 1)$. Since she scored 98 in that test, an alternative way of writing this is $ns + 98$, since her total increased by 98. Equating our two expressions, we know that $ns + 98 = (n + 1)(s + 1)$. Expanding the brackets and subtracting $ns + 1$ from each side gives

$$s + n = 97.$$

After sitting her final test, Ellie has sat $n + 2$ tests, at an average score of $s - 1$, since this average decreased by 2 from the previous one. This means her total score for all her tests is $(n + 2)(s - 1)$. Since she scored 70, her total score can also be written as $ns + 98 + 70 = ns + 168$. Since these two expressions are equal, we know that $ns + 168 = (n + 2)(s - 1)$. Expanding the brackets and subtracting $ns - 2$ from each side gives

$$2s - n = 170.$$

We have obtained two equations connecting n and s. Adding these together gives $3s = 267$, so $s = 89$. This means $n + 89 = 97$, so $n = 8$. This means Ellie had sat 8 tests before her final two, so she sat a total of 10 tests.

5. Three pirates are washed up tired and hungry on a desert island, on which there are some palm trees and a troop of monkeys. Having collected some coconuts from the palm trees, they decide to put these in a pile and share them out in the morning.

In the night, the first pirate wakes up and decides that she cannot trust her friends, so decides to claim her share. She then counts the coconuts and finds that there is one too many to divide equally. Throwing one to the nearest monkey, she buries a third of the remainder.

An hour later, the second pirate wakes up and also decides that he cannot trust his friends. Counting the coconuts, he finds that there is one too many to share equally. Throwing one to the nearest monkey, he buries a third of the remainder.

An hour later, the final pirate wakes up with the same thought. Finding the coconuts do not split equally, she throws one coconut to the nearest monkey, and then buries a third of the remainder.

In the morning, all the pirates keep quiet about their night-time exploits. They split the pile of coconuts equally between them, but there is one left over, which they throw to the nearest monkey.

What is the smallest number of coconuts that they could have started with for this to be possible? (The pirates have no way of cutting coconuts - at all points all the coconuts remain whole.)

Answer 79

Solution

Let us denote by x the number of coconuts that each of the pirates obtained from the splitting of the final pile. We know that this must be an integer, and must be at least 0. This means that, when the pirates woke in the morning, there were $3x + 1$ coconuts in the pile, as they each got x and one was thrown to the monkeys.

Then, we can consider the situation before the final pirate awoke in the night. One coconut was thrown to the monkeys and she took one third of the remaining pile. This means two-thirds of the pile was left, with size $3x + 1$. The original size of the pile was therefore $\frac{3}{2}(3x + 1) + 1$.

Since we know that this is an integer, we also know that $3x + 1$ must be even, so $3x$ must be odd, so x must be odd. Write $x = 2y + 1$, where we know that y is an integer. Since $x \geq 0$ and x was odd, $x \geq 1$, which means that $y \geq 0$. Then, we know that the size of the pile before the third pirate awoke was

$$\tfrac{3}{2}(3x + 1) + 1 = \tfrac{3}{2}(3(2y + 1) + 1) + 1 = \tfrac{3}{2}(6y + 4) + 1 = 9y + 7$$

We can then work back in the same way to the point before the second pirate awoke. There must have been $\frac{3}{2}(9y + 7) + 1$ coconuts in this pile. Again, this is an integer, so $9y + 7$ must be even, which tells us that y must be odd. We can therefore write $y = 2z + 1$, with z an integer at least 0, and the pile has size

$$\tfrac{3}{2}(9y + 7) + 1 = \tfrac{3}{2}(9(2z + 1) + 7) + 1 = \tfrac{3}{2}(18z + 16) + 1 = 27z + 25$$

We can then repeat the same idea for the original pile. There were $\frac{3}{2}(27z + 25) + 1$ coconuts in the pile. As this is an integer, we know that $27z + 25$ must be even, so z is odd. For the pile to be as small as possible, we need z to be as small as possible (as the size of the pile increases as z increases). The smallest possible value of z is 1, as it must be an odd integer greater than or equal to 0.

Then, the pile has size $\frac{3}{2}(27 \times 1 + 25) + 1 = 79$.

We can check that this works as follows.

- The first pirate threw one coconut to the monkeys and buried $\frac{1}{3} \times 78 = 26$. This left 52 coconuts.
- The second pirate threw one coconut to the monkeys and buried $\frac{1}{3} \times 51 = 17$. This left 34 coconuts.
- The third pirate threw one coconut to the monkeys and buried $\frac{1}{3} \times 33 = 11$. This left 22 coconuts.
- The pirates then threw one more coconut to the monkeys, and kept $\frac{1}{3} \times 21 = 7$ coconuts each.

Alternative

Suppose that there are n coconuts in the pile originally. The first pirate throws one to a monkey, and then takes one third. This means that $n - 1$ is divisible by 3, say it is equal to $3p$, where p is an integer and is non-negative. Thus the first pirate leaves $2p$ coconuts in the pile.

The second pirate then throws one to the monkeys, leaving $2p - 1$. For this to be divisible by 3, we need $2p \equiv 1 \pmod 3$. Multiplying by 2 gives $p \equiv 2 \pmod 3$, so p is of the form $3q + 2$. Since $p \geq 0$, $q \geq -\frac{2}{3}$. However, since q is an integer, we must have $q \geq 0$. This means the second pirate buries $2q + 1$ coconuts and the pile left is of size $4q + 2$.

Then the third pirate throws one coconut to the monkeys, leaving $4q + 1$. Since this must be divisible by 3, $4q \equiv 2 \pmod 3$. This in turn implies that $q \equiv 2 \pmod 3$, or q is of the form $3r + 2$ for r an integer. Then, as $q \geq 0$, we can again deduce that $r \geq -\frac{2}{3}$, so $r \geq 0$. This means the third pirate took $4r + 3$ coconuts, leaving $8r + 6$ in the pile.

One is then thrown to the monkeys, leaving $8r + 5$. As this is shared equally between all three pirates, it must be a multiple of three, so $8r \equiv -5 \pmod 3$, or $2r \equiv 1 \pmod 3$. Doubling gives $r \equiv 2 \pmod 3$.

The smallest pile is obtained from the smallest p, which is obtained from the smallest q, which is in turn obtained from the smallest r. The smallest solution is therefore $r = 2$, meaning $q = 8$ and $p = 26$. This means $n = 79$, and we can check this as before.

6. In a large office, each person has their own telephone extension, whose number consists of three digits, but not all possible extensions are in use. In an effort to prevent mistakes in dialling, no number in use can be converted to another just by swapping two of its digits. What is the largest possible number of extensions in use in the office?

Answer 460

Solution

To pick the three digits for an extension, there are three distinct cases that we need to consider. These cases depend on how many of the digits are the same, since swapping digits can never change the numbers, only their order. For each of these cases, we need to consider how many extensions it yields, and how many numbers of this type there are.

- All digits are the same.

  If all the digits are the same, then there is only one possible extension.

To choose the repeated digit, there are ten possibilities.

This gives a total of $1 \times 10 = 10$ numbers.

- Two digits are the same.

  Suppose we have two digits a and one digit b. There are then three numbers that are possible: aab, aba and baa. However, as any of these is a single swap away from any other, at most one can be in use.

  There are ten ways to choose the repeated digit. There are then nine ways to choose the other digit, so there is a total of $10 \times 9 = 90$ ways.

  This gives a total of $1 \times 90 = 90$ extensions.

- All the digits are different.

  Let the digits be a, b and c. This means there are six possible extensions: abc, acb, bac, bca, cab and cba. If we have one extension, we can label the digits so that this is abc. This rules out acb (swapping b and c), cba (a and c) and bac (a and b). This leaves abc, bca and cab. These each differ from each other in every position, so cannot be obtained from each other via a single swap. This means we can obtain 3 extensions from each set of three distinct numbers.

  There are $\binom{10}{3} = \frac{10\times9\times8}{3\times2\times1} = 120$ different ways of choosing the three digits from the set of 10.

  This gives a total of $3 \times 120 = 360$ extensions.

The total number of extensions is therefore $10 + 90 + 360 = 460$.

7. The integers 1 to 1000 are written on a whiteboard. Rhiannon picks a pair of numbers a and b written on the board, rubs them out and replaces them with the number $a + b - 1$. If she keeps doing this until there is just one number left, what possibilities are there for this number?

ANSWER 499501 only

SOLUTION

> At the start of this question it is not at all obvious where to begin. Hence a good idea is to try a few cases with a much smaller set of numbers, say 1 to 4. If you do this you might notice that, whatever you do, you always seem to end up with the same answer, in this case 7. You might think that this would remain the same for the larger set of numbers, but why?
>
> At this point it is a good idea to try and look for something that behaves in the same whatever pair of numbers you combine. Each time a and b are replaced by an expression containing $a + b$: perhaps you should consider the sum of the numbers on the board.

Consider the sum of all the numbers on the board. Every time two numbers a and b are rubbed out, they are replaced by $a + b - 1$, so the sum of the numbers has decreased by 1. Since the number of numbers needs to be reduced from 1000 to 1, and each replacement reduces the number by 1, there are 999 replacements needed.

Therefore we need to calculate the sum of all the 1000 numbers on the board. There are many ways to do this (effectively we want to know the 1000^{th} triangle number), but one neat way is to consider the numbers in pairs: 1 and 1000, 2 and 999, ..., 500 and 501. There are 500 pairs, each with a total of 1001, so the total of all the numbers must be $500 \times 1001 = 500500$.

Then, subtracting 1 for each of the 999 replacements means that the total provided by the one remaining number must be $500500 - 999 = 499501$. Therefore that is the remaining number, and there are no other choices.

Alternative

Imagine re-writing the numbers as $(0+1), (1+1), (2+1), \ldots (999+1)$. Then, when we erase $x+1$ and $y+1$, we replace them with $x+y+1$. Repeating this means that we will replace the whole set of numbers with $(0+1+2+\ldots+999)+1$, so we just need to work out this sum.

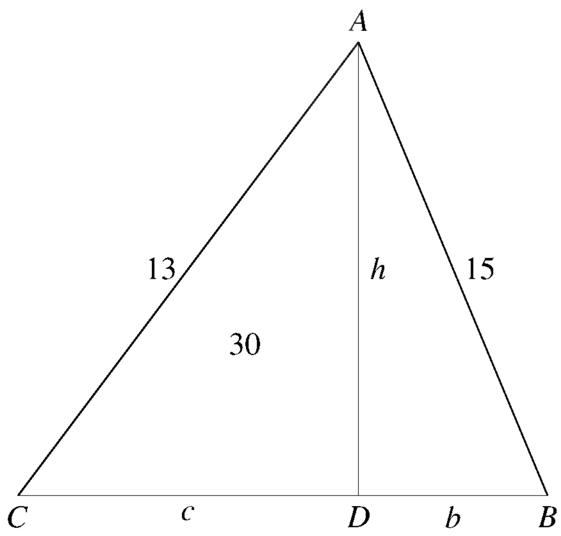
This is the 999^{th} triangle number, and we can pair the numbers up as 0 and 999, 1 and 998, ..., 499 and 500. There are 500 pairs, and each adds to 999, so the total is $999 \times 500 = 499500$.

The means that the number remaining on the board is 499501.

8. Triangle ABC is acute-angled, with $AB = 15$ and $AC = 13$. The point D is the foot of the perpendicular from A to BC, and the area of triangle ADC is 30. If the area of ABC is an integer, then what is this integer?

Answer 84

Solution



Let us denote the length AD by h, the length DB by b and the length CD by c.

Then let us consider the right-angled triangle ADC. Since it has area 30, we know that $\frac{1}{2}ch = 30$, so

$$ch = 60.$$

We can also use Pythagoras' theorem to tell us that

$$c^2 + h^2 = 169.$$

Now we need to calculate the values of c and h. If we add twice the first equation to the second, then we obtain $c^2 + 2ch + h^2 = 289$. This factorises to give $(c+h)^2 = 17^2$. Then $c+h$ is either 17 or -17. However, as both c and h are lengths, we know they are positive, so $c + h = 17$.

If we subtract twice the first equation from the second, we get $c^2 - 2ch + h^2 = 49$. This factorises to give $(c-h)^2 = 49$. This means $c - h$ is either 7 or -7.

Then, adding these results gives $2c = 24$ or $2c = 10$, so $c = 12$ or $c = 5$. This means $h = 5$ or $h = 12$.

Now, using Pythagoras' theorem on triangle ADB, we get that $b^2 = 225 - h^2$. This means b^2 is either 200 if $h = 5$ or 81 if $h = 12$. Next, the area of ADB, which we know must be an integer, is equal to $\frac{1}{2}bh$. This expression is either $\frac{5}{2}\sqrt{200} = 25\sqrt{2}$ or 54. Only the second of these is an integer, which makes the area of triangle ABC equal to $30 + 54 = 84$.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by Oxford Asset Management

Mary Cartwright

Sheet 6

Solutions and comments

This programme of the Mentoring Scheme is named after Dame Mary Lucy Cartwright (1900–1998).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Cartwright.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

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1. Find the smallest number that appears in all these arithmetic sequences.

$$22, 33, 44, 55, \ldots$$
$$24, 37, 50, 63, \ldots$$
$$25, 39, 53, 67, \ldots$$

ANSWER 2013

SOLUTION

Let us consider the general term of each sequence. Since these are linear sequences, the general term is of the form $t_n = an + c$, where a and c are constants. Note that a is the difference between successive terms. Setting $n = 0$ implies that $t_0 = c$ and we can consider this as an initial term before the first term given. Using this we can see that the general terms are as follows.

$$\begin{aligned} r_l &= 11l + 11 \\ s_m &= 13m + 11 \\ t_n &= 14n + 11 \end{aligned}$$

For a number to be in the first sequence, we can see that it needs to be a multiple of 11 more than 11. To be in the second sequence, it needs to be a multiple of 13 more than 11. To be in the third sequence, it needs to be a multiple of 14 more than 11.

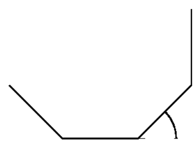
The lowest common multiple of 11, 13 and 14 is 2002. This means that the smallest number in all three sequences is 2013.

2. A polygon with n sides has exactly three obtuse angles, and no reflex angles. Find all possible values of n.

ANSWER 4, 5 or 6 only

SOLUTION

We have three obtuse angles and all the remaining angles must be acute (or right-angled). It will help us to consider the external angles of the polygon:

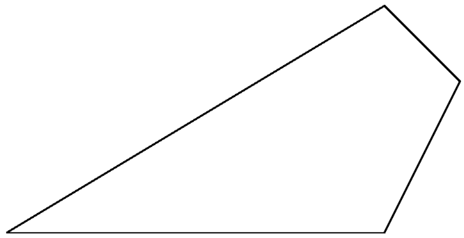


If you walk around the outside of the polygon, then you make one full turn. Therefore the sum of all the external angles is 360°. If x is one of the obtuse angles, then the external angle is $180° - x$. Since x is obtuse, we know that $0° < 180° - x < 90°$. All the other angles are acute or right-angled, so their external angles are at least 90°.

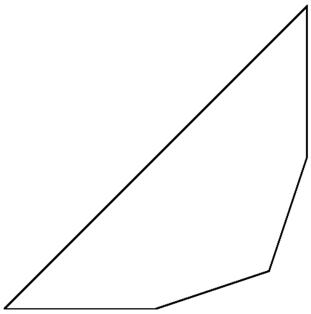
The sum of the angles external to the three obtuse angles must be more than 0° and less than 270°. This leaves more than 90° and less than 360° for the external angles of the remaining angles. This means there is at least one and at most three acute or right angles.

Consequently, $n = 4$, $n = 5$ or $n = 6$. The following diagrams show that we can indeed construct each of these polygons.

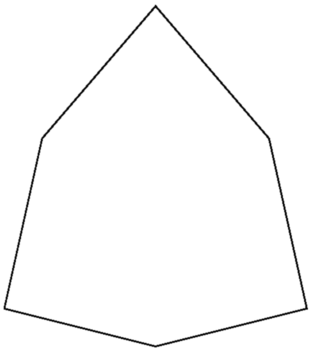
$n = 4$



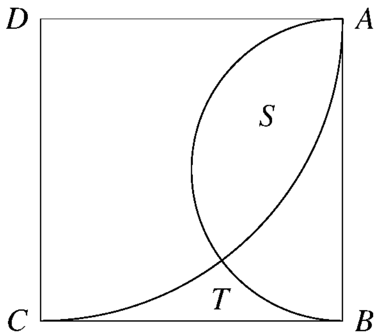
$n = 5$



$n = 6$



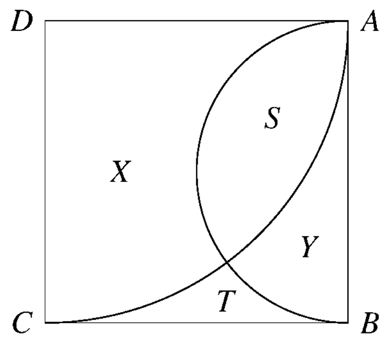
3. The square $ABCD$ shown has side length $2\sqrt{2}$. A semicircle with diameter AB and a quarter-circle with centre D, passing through A and C, have been added. The areas of the two marked regions are S and T. What is the value of $S - T$?



Answer $3\pi - 8$

Solution

Let us label the other portions of the square as X and Y.



Since the square has side length $2\sqrt{2}$, this is the radius of the quarter circle. Therefore, the quarter-circle has area

$$X + S = \frac{1}{4}\pi\left(2\sqrt{2}\right)^2 = 2\pi.$$

The semicircle has radius half the side of the square, which is $\sqrt{2}$. This means that the area of the semicircle is

$$Y + S = \frac{1}{2}\pi\left(\sqrt{2}\right)^2 = \pi.$$

The square has side length $2\sqrt{2}$, so has area

$$S + T + X + Y = \left(2\sqrt{2}\right)^2 = 8.$$

To get the $-T$, we first need to subtract the area of the square. We then have a $-X$ and a $-Y$, so we need to add in the areas of the semicircle and quarter-circle. This means we have

$$S - T = (S + X) + (S + Y) - (S + T + X + Y) = 2\pi + \pi - 8 = 3\pi - 8.$$

4. What is the largest two-digit prime number which is a factor of $\binom{200}{100}$?

Remember that $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

Answer 61

Solution

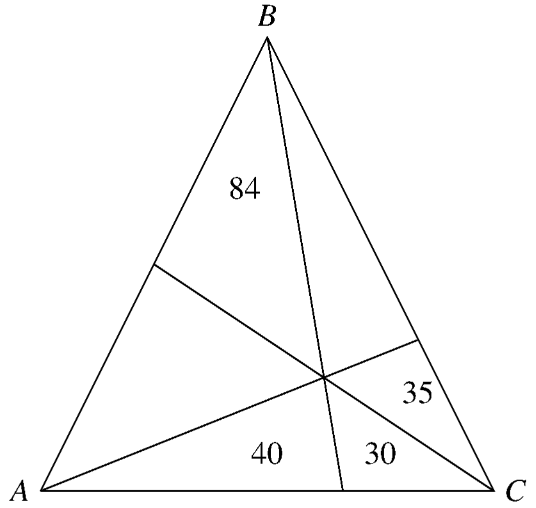
Let us consider a prime p, and see if it divides $\binom{200}{100}$. We want p to have two digits, so we need to consider only cases where $p \leq 99$. Remember that $\binom{200}{100} = \frac{200!}{100!100!}$.

If $67 \leq p \leq 99$, then $p \leq 100 < 2p \leq 200 < 3p$. This means that p appears in the product for 100! but $2p$ does not. Also, p and $2p$ appear in that for 200!, but $3p$ does not. This means that $\binom{200}{100} = \frac{200!}{100!100!}$ has exactly two factors of p in the numerator and exactly two in the denominator. Thus it does not have a factor of p.

If $51 \leq p \leq 66$, then $p \leq 100 < 2p < 3p \leq 200 < 4p$. This means that 200! has exactly three factors of p. However, 100! has exactly one factor of p, so $\binom{200}{100} = \frac{200!}{100!100!}$ has exactly three factors of p in the numerator, and two in the denominator. This means p divides $\binom{200}{100}$.

The largest such prime is 61, so 61 is the largest two-digit prime dividing $\binom{200}{100}$.

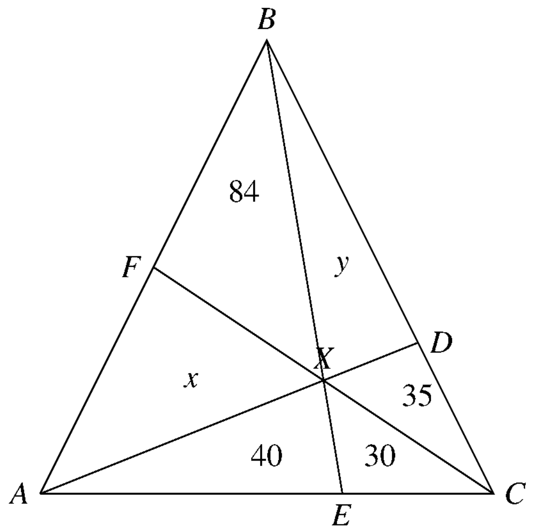
5. The triangle below has been divided into six regions by three lines, all of which pass through the same point. Four of the regions have been labelled with their areas. What is the area of triangle ABC?



Answer 315

Solution

Let's put some more labels on the diagram, so that we can talk about all the different areas involved.



In this solution, we will repeatedly use the following fact. If two triangles have their bases on the same line and also have the same vertex, then they have the same height. It follows that the ratio of their areas is the same as that of their bases.

Triangles AXE and EXC have bases on the same line AC and X is their common vertex. This tells us that $AE : EC = [AXE] : [EXC] = 40 : 30 = 4 : 3$. Note that here we use a standard square-bracket notation for the area of a polygon. For example, $[AXE]$ denotes the area of triangle AXE.

The same argument applies to triangles ABE and EBC: these have bases on the line AC and common vertex B. This tells us that $AE : EC = [ABE] : [EBC]$. Putting this together gives

$$x + 124 : y + 65 = [ABE] : [EBC] = AE : EC = 4 : 3.$$

We can repeat this argument using the line BC as the base. With X as the vertex,

$$BD : DC = [BXD] : [DXC] = y : 35.$$

With A as the vertex,

$$BD : DC = [BAD] : [DAC] = x + y + 84 : 105.$$

Therefore

$$y : 35 = x + y + 84 : 105.$$

We now have equivalent ratios, which we can convert into a pair of simultaneous equations:

$$\begin{aligned} \frac{x+124}{y+65} &= \frac{4}{3}; \\ \frac{x+y+84}{105} &= \frac{y}{35}. \end{aligned}$$

Clearing the denominators of these equations gives:

$$\begin{aligned} 3x + 372 &= 4y + 260; \\ x + y + 84 &= 3y. \end{aligned}$$

Simplifying these equations gives:

$$\begin{aligned} 3x + 112 &= 4y; \\ x + 84 &= 2y. \end{aligned}$$

Subtracting twice the second equation from the first tells us that $x - 56 = 0$, so $x = 56$. Then $y = \frac{1}{2}(56 + 84) = 70$.

This means that the whole area of the triangle is $84 + 70 + 56 + 35 + 40 + 30 = 315$.

6. Rhiannon has once again written the integers between 1 and 1000 on her whiteboard. This time, when she rubs out the numbers a and b, she replaces them with $ab + a + b$. If she keeps doing this until just one number is left, what possibilities are there for this final number?

ANSWER $1001! - 1$

SOLUTION

Rhiannon replaces a and b with $ab + a + b$. This can be rewritten as $(a + 1)(b + 1) - 1$. We can also re-write a and b as $(a + 1) - 1$ and $(b + 1) - 1$. This suggests that we ought to consider what happens to the numbers one larger than those written on the board.

At each step, these "one-larger" numbers multiply. This means that it does not matter in which order we combine the numbers, we will end up with the "one-larger" product of $(1 + 1)(2 + 1)\dots(1000 + 1) = 2 \times 3 \times \dots \times 1001 = 1001!$

This means that the number that Rhiannon ends up with is $1001! - 1$, no matter in what order she carries out her operations.

> There is no need to try and work out the value of $1001! - 1$, as this number has 2005 digits.

7. Eight differently coloured triangles are put together to form a regular octahedron. If two arrangements are considered to be the same if one can be rotated to form the other, how many different regular octahedra can be made?

ANSWER 1680

SOLUTION

If we treat the different orientations as different, then there are $8!$ ways of colouring the faces of the regular octahedron. This follows because, if we order the faces, then there are $8!$ corresponding ways of ordering the colours.

However, we can then count the number of ways in which each of these regular octahedra can be rotated. There are eight different faces that could be arranged to face upwards. Moreover, there are three rotations of each face since each is an equilateral triangle. Altogether this gives a total of 24 rotations of the same shape.

Therefore, we have counted each colouring 24 times, so the number of distinct colourings is $\frac{8!}{24} = 1680$.

ALTERNATIVE

We can always rotate the regular octahedron so that a particular colour, say red, is on top.

Then, there are $\binom{7}{3} = 35$ ways of choosing the three colours that will be on the adjacent faces, let us call them A, B and C. However, there are two arrangements of these: either they can be ABC counting clockwise, or ACB.

After this, there are four remaining colours that could touch both A and B, three for A and C and two for B and C. This leaves the last colour to go on the bottom. There are then no ways left of rotating the regular octahedron.

This means that the total number of colourings is $35 \times 2 \times 4 \times 3 \times 2 \times 1 = 1680$.

8. Suppose A is a subset of $\{1, 2, \ldots, n\}$. The *alternating sum* of A is defined by arranging the elements of A in descending order, then alternately adding and subtracting successive numbers. For example, the alternating sum of $\{1, 2, 4, 6, 9\}$ is $9 - 6 + 4 - 2 + 1 = 6$ and for $\{5\}$ it is simply 5.

What is the sum of the alternating sum of all subsets of $\{1, 2, 3, 4, 5, 6, 7\}$?

ANSWER 448

SOLUTION

> A useful observation for this question is that in any subset of $\{1, 2, 3, 4, 5, 6, 7\}$, 7 is always the largest number (if it occurs), so always has a plus sign in the alternating sums. All the other numbers will sometimes have an odd number of numbers larger than them and sometimes an even number. Thus they will occur with different signs in the sums for different sets.

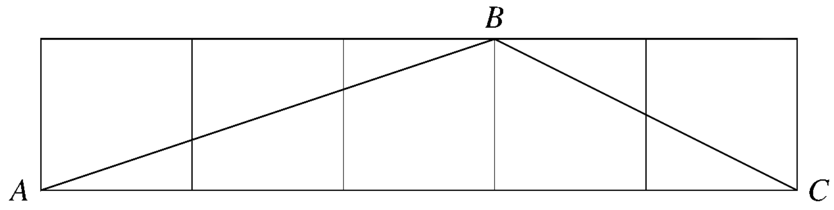
Suppose that $S \subseteq \{1, 2, 3, 4, 5, 6\}$ (which is a subset that does not contain a 7); say $S = \{s_1, s_2, \ldots s_k\}$. Then, the alternating sum is $s_1 - s_2 + \ldots \pm s_k$, where we are assuming that the s_i are in descending order.

Now consider the set $S \cup \{7\}$, where we have added a 7 to S. Then $S \cup \{7\} = \{7, s_1, s_2, \ldots s_k\}$ is still in descending order; furthermore, it has alternating sum $7 - s_1 + s_2 - \ldots \mp s_k$.

Therefore, pairing together these two sets which differ only by the inclusion of a 7, the total of their alternating sums is exactly 7. This is because every number with a plus sign in the alternating sum for S has a minus sign in that for $S \cup \{7\}$ and vice-versa.

There are $2^6 = 64$ subsets of $\{1, 2, 3, 4, 5, 6\}$, as we can independently choose to include or not include each of the six numbers. This means there are 64 pairs of the form above, each of which contributes 7 to the total of all the alternating sums. This means that the total of the alternating sums is $64 \times 7 = 448$.

9. In the diagram below, five identical squares have been placed together. What is the angle $\angle ABC$?



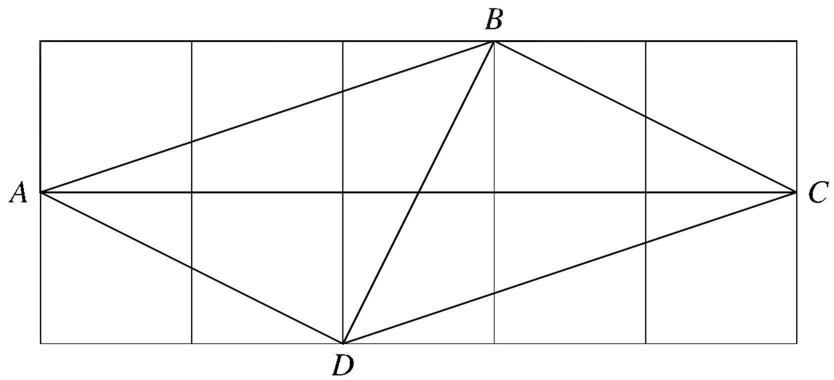
Answer 135°

Solution

It is not immediately obvious how to start this question. One thing that might be useful to notice is the lengths of AB and BC. If the side-length of the squares is scaled to be 1, then Pythagoras' theorem tells us that:

$$\begin{aligned} AB &= \sqrt{1^2 + 3^2} = \sqrt{10}; \\ BC &= \sqrt{1^2 + 2^2} = \sqrt{5}. \end{aligned}$$

This tells us that $AB = \sqrt{2}BC$, so these could form the side lengths of an isosceles right-angled triangle.



In this diagram, we have extended the grid formed by the squares and added the vertex D.

Then, consider what happens if we rotate the line BC by 90° clockwise about B. The two squares between D and B rotate on the grid to give the two squares between D and A.

Therefore, the line DB rotates to become the line DA. This tells us that $\angle DBC = 90°$.

In a similar manner, consider rotating the line DB by 90° anticlockwise around D. This will end up being the line DA, which can be seen from the grid. This tells us both that $\angle ADB = 90^\circ$, and also that $AD = DB$. Therefore ADB is an isosceles triangle, so $\angle DAB = \angle DBA$. Each of these angles is therefore $\frac{1}{2}(180^\circ - \angle ADB) = \frac{1}{2}(180^\circ - 90^\circ) = 45^\circ$.

Finally, we can calculate the angle $\angle ABC = \angle ABD + \angle DBC = 45^\circ + 90^\circ = 135^\circ$.

United Kingdom
Mathematics Trust

Mentoring Scheme

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Mary Cartwright

Sheet 7

Solutions and comments

This programme of the Mentoring Scheme is named after Dame Mary Lucy Cartwright (1900–1998).

See http://www-groups.dcs.st-and.ac.uk/history/Biographies/Cartwright.html for more information.

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1. Determine all triples (a, b, c) of integers such that $1 \le a \le b \le c$ and

$$\frac{1}{a+2} + \frac{1}{b+2} = \frac{1}{2} + \frac{1}{c+2}.$$

Answer $(1, 1, 4)$, $(1, 2, 10)$ and $(1, 3, 28)$

Solution

Since we know that

$$\frac{1}{a+2} + \frac{1}{b+2} = \frac{1}{2} + \frac{1}{c+2} > \frac{1}{2},$$

it follows that at least one of $\frac{1}{a+2}$ and $\frac{1}{b+2}$ is greater than $\frac{1}{4}$. We are told in the question that $a \le b$, so

$$\frac{1}{a+2} \ge \frac{1}{b+2}$$

we must have $\frac{1}{a+2} > \frac{1}{4}$. This tells us that $a + 2 < 4$, so $a < 2$. But a is an integer and $a \ge 1$, so $a = 1$.

Substituting for a yields

$$\frac{1}{3} + \frac{1}{b+2} = \frac{1}{2} + \frac{1}{c+2}.$$

Simplifying gives

$$\frac{1}{b+2} = \frac{1}{6} + \frac{1}{c+2}.$$

Since this implies $\frac{1}{b+2} < \frac{1}{6}$, $b < 4$. This gives the following three options.

- If $b = 1$, then $\frac{1}{c+2} = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$, so $c = 4$.
- If $b = 2$, then $\frac{1}{c+2} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$, so $c = 10$.
- If $b = 3$, then $\frac{1}{c+2} = \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$, so $c = 28$.

Finally, the relevant substitutions show that each of the triples found is a valid solution.

2. (a) Can you expand the brackets in $(a + b)(a - b)$?

(b) When each digit of a four-digit perfect square is increased by 3, the result is another four-digit perfect square. Find all four-digit perfect squares that have this property.

Answer

(a) $(a + b)(a - b) = a^2 - b^2$

(b) The only square is $34^2 = 1156$, with $4489 = 67^2$.

Solution

(a) $(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2$

> This is called the *difference of two squares* factorisation, because of the form that the right-hand side takes. It is a useful tool for solving lots of algebra problems, such as part (b)!

(b) Let us call the two square numbers x^2 and y^2. Since these are both four-digit numbers, we know that $1000 \le x^2, y^2 < 10000$, so $31 < \sqrt{1000} \le x, y < 100$.

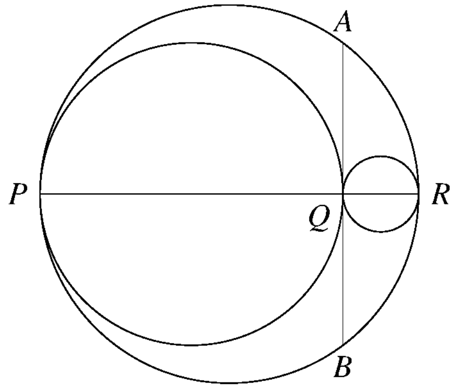
Since each of the digits is to increase by 3, we have $x^2 + 3333 = y^2$. This can then be rearranged to give $y^2 - x^2 = 3333$. Then, we can use the difference of two squares factorisation from part (a) to give $(y + x)(y - x) = 3333$.

Since $x < y$, we know that $0 < y - x < y + x < 200$, so we are looking for two positive integers less than 200 which have a product of 3333. As a product of prime factors, $3333 = 3 \times 11 \times 101$. As 101 is more than half of 200, we know that this must be on its own as a factor. The other factor is $3 \times 11 = 33$. Therefore:

$$\begin{aligned} y + x &= 101 \\ y - x &= 33 \end{aligned}$$

Adding and halving gives $y = 67$, so $x = 34$. This means that the square numbers are $34^2 = 1156$ and $67^2 = 4489$, which have the correct difference.

3. The points P, Q and R lie on a line. Three circles, with diameters PQ, PR and QR are drawn and the chord AB is drawn such that it is tangent to the other two circles at Q, as shown. The line AB has length 8.

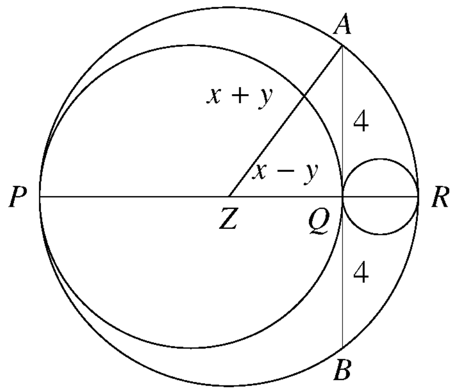


What is the area inside the large circle but outside the small circles?

ANSWER 8π

SOLUTION

Let the three circles have centres X, Y and Z respectively (see the diagram below). Then, if the radii of the smaller circles are x and y, we know that the large circle has radius $x + y$, as the diameters add. Since PQ has length $2x$ and PZ has length $x + y$, this means ZQ has length $x - y$. (See the diagram again.)



Now, we also know that AB is tangent to the inner circles at Q, so it is perpendicular to the radii of the circles. This means that $\angle ZQA$ is a right angle, so ZQA is a right angled triangle.

Then Pythagoras' theorem tells us that

$$(x+y)^2 = (x-y)^2 + 4^2.$$

Expanding the brackets and collecting like terms gives $4xy = 16$, or $xy = 4$.

The area of the whole large circle is $\pi(x+y)^2$ and those of the smaller circles are πx^2 and πy^2. Therefore the leftover area is

$$\begin{aligned}\pi(x+y)^2 - \pi x^2 - \pi y^2 &= x^2\pi + 2xy\pi + y^2\pi - x^2\pi - y^2\pi \\ &= 2xy\pi \\ &= 8\pi\end{aligned}$$

> It is interesting to notice in this solution that we do not at any stage calculate the values of x and y. In fact, you can imagine making the outer circle larger, and therefore moving the chord AB nearer to R to ensure that it still has the same length. This will make the area in question remain the same, even though its shape will have changed!

4. If each of a, b, c and d has value 1, 2 or 3 and

$$a - 3b - 9c + 27d = 19,$$

then what are the values of a, b, c and d?

ANSWER $a = 1$, $b = 3$, $c = 3$, $d = 2$

SOLUTION

> Since we are told that each of the values is either 1, 2 or 3, this means that it is enough for us to know the remainder when each of the values is divided by 3 - in other words their values modulo 3. We can use this repeatedly to help us solve the problem.

Let us consider the equation modulo 3. It becomes

$$a - 3b - 9c + 27d \equiv 19 \pmod 3.$$

This simplifies to become $a \equiv 1 \pmod 3$, so this tells us that $a = 1$.

Then, we can substitute this in to obtain

$$-3b - 9c + 27d = 18.$$

Dividing by -3 then gives

$$b + 3c - 9d = -6.$$

Reducing this modulo 3 tells us that $b \equiv 0 \pmod 3$, so we know that $b = 3$.

Substituting this into the previous equation, we get

$$3c - 9d = -9.$$

Dividing by 3 yields

$$c - 3d = -3,$$

so reducing modulo 3 gives $c \equiv 0 \pmod 3$. Hence $c = 3$.

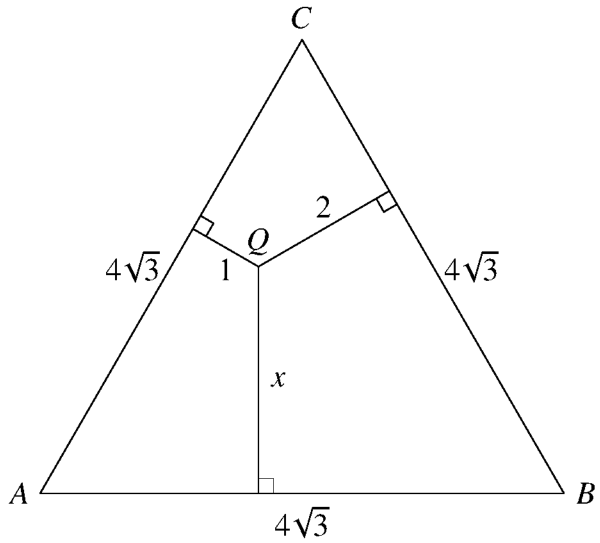
Finally, this tells us that $9 - 9d = -9$, which rearranges to give $d = 2$. This means that the only possible solution is $a = 1$, $b = 3$, $c = 3$ and $d = 2$. Substituting this in the original equation confirms that it works.

5. An equilateral triangle has side length $4\sqrt{3}$. A point Q is situated inside the triangle so that its perpendicular distances from two of the sides of the triangle are 1 and 2. What is the perpendicular distance to the third side?

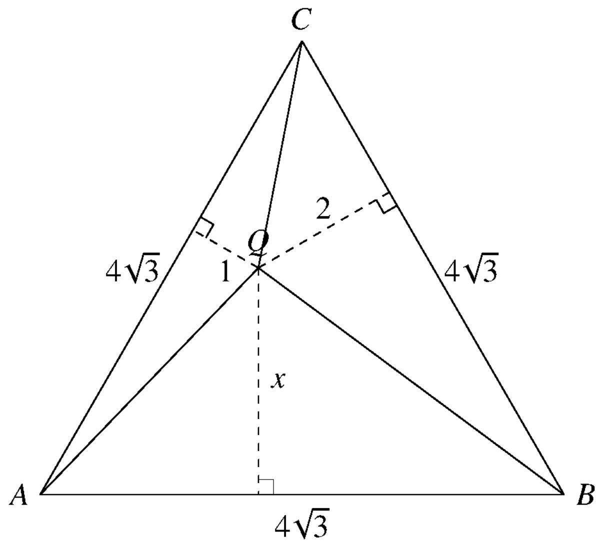
Answer 3

Solution

Drawing a diagram for the problem shows the following configuration.

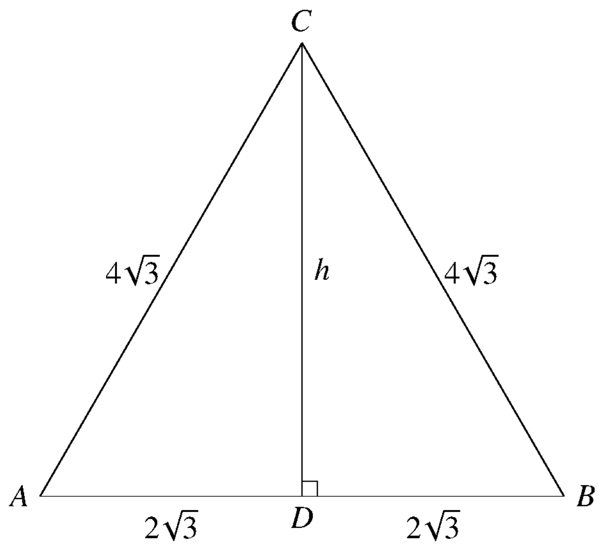


Next, we add the lines AQ, BQ and CQ to the diagram; the triangle is now divided up into three smaller triangles.



Using the formula Area $= \frac{1}{2} \times$ base $\times$ height, we find that the area of triangle AQC is $\frac{1}{2} \times 4\sqrt{3} \times 1 = 2\sqrt{3}$. In the same way, the area of BQC is $\frac{1}{2} \times 4\sqrt{3} \times 2 = 4\sqrt{3}$ and that of AQB is $\frac{1}{2} \times 4\sqrt{3} \times x = 2x\sqrt{3}$.

We can also calculate the area of the whole triangle ABC. To do this, we need to find the height of the triangle, marked as h in the diagram below.



Now, using Pythagoras' theorem on triangle ACD, we see that

$$h = \sqrt{\left(4\sqrt{3}\right)^2 - \left(2\sqrt{3}\right)^2} = \sqrt{48 - 12} = \sqrt{36} = 6.$$

This means that the area of the whole triangle is

$$\tfrac{1}{2} \times 4\sqrt{3} \times 6 = 12\sqrt{3}.$$

Since the three smaller triangles AQC, AQB and BQC have the same total area as the large triangle, this tells us that the following equality holds:

$$12\sqrt{3} = 2\sqrt{3} + 4\sqrt{3} + 2x\sqrt{3}.$$

Rearranging this equation gives $x = 3$, which is the length that we wanted.

> Interestingly, the total length of the three perpendiculars to any point in the equilateral triangle always remains the same, wherever in the triangle the point is! Can you see how to use the same method as this to prove it?

6. Determine all integers n such that $\sqrt{n - 4\sqrt{n-19}}$ is also an integer.

Remember that $\sqrt{x}$ denotes the positive square root of x.

Answer $n = 20$, $n = 28$ or $n = 100$

Solution

Let us call the value of the given expression x, which we know must be an integer. Squaring the expression then tells us that

$$x^2 = n - 4\sqrt{n-19}.$$

Since x and n are both integers, this tells us that $4\sqrt{n-19}$ must also be an integer, say k.

But we can rearrange this to obtain $n = 19 + \left(\frac{k}{4}\right)^2$. Consequently k must be a multiple of 4, say $k = 4m$. Hence $m = \sqrt{n-19}$ or $n = m^2 + 19$. Substituting in the first equation yields

$$x^2 = m^2 + 19 - 4m.$$

On the right-hand side we can complete the square to obtain

$$x^2 = (m-2)^2 + 15.$$

In this last equation we have two squares which have a difference of 15. Using the difference of two squares factorisation from Question 2, we find that

$$(x + m - 2)(x - m + 2) = 15.$$

Since $x + m - 2$ and $x - m + 2$ are both integers, this must be one way of factorising 15. This means it must be one of the sums $15 \times 1, 5 \times 3, 3 \times 5, 1 \times 15, -1 \times -15, -3 \times -5, -5 \times -3, -15 \times -1$. The table below shows each of these options.

$x + m - 2$	$x - m + 2$	x	m	n
15	1	8	9	100
5	3	4	3	28
3	5	4	1	20
1	15	8	-5	44
-1	-15	-8	9	100
-3	-5	-4	3	28
-5	-3	-4	1	20
-15	-1	-8	-5	44

However, we know that both $x = \sqrt{n - 4\sqrt{n-19}}$ and $m = \sqrt{n-19}$ are the positive square roots of positive integers, so are positive. This means that we can exclude the lower five lines in the table, leaving the only possible values for n as 20, 28 and 100.

We can check that each of these gives a valid solution by substitution into the original expression. Doing so gives the values 4, 4 and 8, which are all integers as required.

7. Given that

$$20! = 243290a0081766bc000$$

determine the digits a, b and c. You don't need a calculator!

ANSWER $a = 2, b = 4, c = 0$

SOLUTION

This question is all about the various division rules that you might have met. In particular, the following divisibility rules will be useful to us:

- A number is divisible by 10 exactly when it ends in a 0.
- A number is divisible by 8 exactly when the number formed by the last three digits is divisible by 8.
- A number is divisible by 9 exactly when the sum of the digits is divisible by 9.

If you have not seen these rules before, then it is an interesting exercise to try and explain why

‖ all three are true. ‖

From the question, we know that

$$20\times19\times18\times17\times16\times15\times14\times13\times12\times11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1 = 243290a0081766bc000.$$

Since the number on the right-hand side ends in three 0's, it has a factor of 1000. Dividing both sides by 1000 (and using the fact that $1000 = 20 \times 10 \times 5$) yields

$$19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 9 \times 8 \times 7 \times 6 \times 4 \times 3 \times 2 = 243290a0081766bc$$

Now, the left-hand side has a factor of 15 and so a factor of 5; it also has a factor of 2. This means that it is divisible by 10, so the right-hand side is also divisible by 10. This can occur only if it ends in a 0, so $c = 0$.

Dividing both sides by this factor of 10,

$$19 \times 18 \times 17 \times 16 \times 3 \times 14 \times 13 \times 12 \times 11 \times 9 \times 8 \times 7 \times 6 \times 4 \times 3 = 243290a0081766b$$

The left-hand side is clearly divisible by 8. This means the right-hand side is also divisible. Therefore (see the rules above), the last three digits must form a number that is divisible by 8. These last digits are $66b$. The only possible multiple of 8 is $664 = 8 \times 83$, so we know that $b = 4$.

The left-hand side has a factor of 9, so the right-hand side must have one also. This means that the sum of the digits is a multiple of 9. Now $2 + 4 + 3 + 2 + 9 + 0 + a + 0 + 0 + 8 + 1 + 7 + 6 + 6 + 4 = 52 + a$ must be a multiple of 9. The only possible value for a to make this the case is $a = 2$.

This means

$$20! = 2432902008176640000.$$

8. (a) What is the largest set of integers that you can have such that no three of them sum to a multiple of 3?

Remember this statement means that, if you believe the answer is n, then you need to find a set of n such integers. You also need to prove no such set of size n + 1 or more can exist.

(b) Show that given any seventeen integers, it is possible to choose five whose sum is divisible by 5.

(c) *Hard Extension.*

What is the largest set of integers that you can have so that no five of them sum to a multiple of 5?

You know from part (b) that this is at most 16. The question is, how much better than this can you do?

Answer

(a) Size 4, e.g. $\{0, 0, 1, 1\}$

(c) See below - I don't want to spoil it!

Solution

(a) The first thing to notice with this question is that we can work modulo 3, as we are only worried about when the sum is divisible by 3. This means that we only need to count how many 0's, 1's and 2's are in our set.

It is worth noticing that in the set we cannot have the same modulo 3 value repeated three times, as if we have three a's, then $a + a + a = 3a \equiv 0 \pmod 3$, which we did not want. Also, if we have one of

each remainder, then $0 + 1 + 2 = 3 \equiv 0 \pmod 3$, which we also did not want. This means that we can have at most two of the remainders, and at most two of each, so a total of four numbers in the set.

We can make a set of four numbers such as $\{0, 0, 1, 1\}$ where no three of them sum to $0 \pmod 3$. The sum of any set of three of these can only be 1 or 2, so this set suffices.

(b) Following the previous part of the question, we can work modulo 5. Again, if we have five numbers with the same remainder, say a, then these have a total congruent to $5a \equiv 0 \pmod 5$, which we did not want. Again, if we have all five remainders present, then the total is $0 + 1 + 2 + 3 + 4 \equiv 10 \equiv 0 \pmod 5$, which is a multiple of 5. This means that we can have at most four of the five remainders present, and at most four of each of these, so a total of at most $4 \times 4 = 16$ numbers in the set. Make sure you understand why this answers part (b).

> These comments suggest a set $\{0, 0, 0, 0, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3\}$, for example. But then $0+0+0+2+3 \equiv 5 \equiv 0 \pmod 5$, which we wanted to avoid. This observation is relevant to the question asked in part (c). How much better can we do?

(c) There are a couple of useful observations that we can make.

- Suppose we have a set containing a subset whose total is divisible by 5. If we add a constant to all the numbers in the set, then the subset still has a total divisible by 5. On the other hand, if the set has no subset with a total divisible by 5, then that property still holds after the addition of a constant to each number in the set.

  Suppose that we add k to each number and these numbers are a, b, c, d and e. Then the new total is $(a + k) + (b + k) + (c + k) + (d + k) + (e + k) = a + b + c + d + e + 5k \equiv a + b + c + d + e \pmod 5$. This means the remainder of the total for the subset does not change.

- Similarly, we can multiply all the numbers in the set by any number that is not divisible by 5, without changing the property that we are interested in.

  Suppose we multiply all the numbers in our set by j and suppose that our original subset is $\{a, b, c, d, e\}$. This becomes $\{ja, jb, jc, jd, je\}$ with total $j(a + b + c + d + e)$. Since 5 is a prime, and j is not divisible by 5, this total is divisible by 5 exactly when $a + b + c + d + e$ is.

Now, from part (a) we have a useful set to consider, namely $\{0, 0, 0, 0, 1, 1, 1, 1\}$. Any subset of this of size five has at least one and at most four ones, with all the other elements being 0's. This means the total is between 1 and 4, so is not a multiple of 5.

> This set is an interesting example. It's worth noticing that adding any number to it creates a subset of size five with a sum divisible by 5. This suggests that this *might* be the largest size of set. We will see what happens when we try a set with nine elements.

Suppose that we have a set with nine elements in it. We can have at most 4 different remainders modulo 5 in our set. Since $9 > 2 \times 4$, this means that there must be some remainder a that is repeated at least three times in our set.

However, we can transform our set using the first result above to make sure $a = 0$: we can do this by adding $-a$ to all of the numbers in the set. This means we have at least three 0's in our set.

Since having five 0's (mod 5) would give a subset with a zero remainder total, this means that there are at most four 0's in our set. This leaves five other numbers to be sought. Then, since there are at most three other remainders in the set, one of them must be repeated at least once, say this remainder is b.

But then, since $1 \times 1 \equiv 1 \pmod 5$, $2 \times 3 \equiv 1 \pmod 5$ and $4 \times 4 \equiv 1 \pmod 5$, we can multiply by the appropriate remainder (using the second of our two results) to produce a set with at least two 1's in it.

Our set is now of the form

$$\{0, 0, 0, 1, 1, ?, ?, ?, ?\}.$$

We cannot have a 4, as this would allow the subset $\{0, 0, 0, 1, 4\}$, which has a total of $0 \pmod 5$. Neither can we have a 3, as this would allow a subset of $\{0, 0, 1, 1, 3\}$, again with a total of $0 \pmod 5$.

Furthermore, we cannot add more than one 2 to this set: doing so would allow the subset $\{0, 0, 1, 2, 2\}$. Again, we cannot add more than two 1's, without allowing $\{1, 1, 1, 1, 1\}$; more than one 0 would allow $\{0, 0, 0, 0, 0\}$. This means that the only possibility for the extra numbers is to add 0, 1, 1 and 2. This would produce the set

$$\{0, 0, 0, 0, 1, 1, 1, 1, 2\}.$$

However, this has the subset $\{0, 1, 1, 1, 2\}$, which also has a sum of $0 \pmod 5$.

This means that any set of size nine has a subset of size five which has a sum divisible by 5. Hence we can have at most eight integers in our set. The example above achieves this, so eight is certainly the maximum.

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Mary Cartwright

Sheet 8

Solutions and comments

This programme of the Mentoring Scheme is named after Dame Mary Lucy Cartwright (1900–1998).

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1. In the sum below, each of the letters stands for a different digit. Can you find *all* the solutions? You should prove that you have found them all.

$$\begin{array}{rcc} & T & E \\ + & T & E \\ + & T & E \\ \hline & A & T \\ \hline \end{array}$$

ANSWER
$$\begin{array}{rcc} & 1 & 7 \\ + & 1 & 7 \\ + & 1 & 7 \\ \hline & 5 & 1 \\ \hline \end{array} \qquad \begin{array}{rcc} & 2 & 4 \\ + & 2 & 4 \\ + & 2 & 4 \\ \hline & 7 & 2 \\ \hline \end{array} \qquad \begin{array}{rcc} & 3 & 1 \\ + & 3 & 1 \\ + & 3 & 1 \\ \hline & 9 & 3 \\ \hline \end{array}$$

SOLUTION

We know that $3T < 10$, as the left hand column does not carry. This means that the only possible values of T are 0, 1, 2 and 3.

In each case, we know that $3E \equiv T \pmod{10}$, so, multiplying by 7, we get $E \equiv 7T \pmod{10}$. We can then consider the four possibilities.

$T = 0$: $E \equiv 7 \times 0 \equiv 0 \pmod{10}$, so $E = 0$. Since E and T are different, this cannot occur.

$T = 1$: $E \equiv 7 \times 1 \equiv 1 \pmod{10}$, so $E = 7$. Then the sum is $17 + 17 + 17 = 51$, so $A = 5$.

$T = 2$: $E \equiv 7 \times 2 \equiv 4 \pmod{10}$, so $E = 4$. Then the sum is $24 + 24 + 24 = 72$, so $A = 7$.

$T = 3$: $E \equiv 7 \times 3 \equiv 1 \pmod{10}$, so $E = 1$. Then the sum is $31 + 31 + 31 = 93$, so $A = 9$.

2. How many numbers are factors of 10^{1001} but not of 10^{1000}?

ANSWER 2001

SOLUTION

The prime factorisation of 10^{1000} is $2^{1000} \times 5^{1000}$, and that of 10^{1001} is $2^{1001} \times 5^{1001}$. This means that we need consider only factors of the form $2^a \times 5^b$.

This number will be a factor of 10^{1001} if $0 \leq a \leq 1001$ and $0 \leq b \leq 1001$. It is also a factor of 10^{1000} if $0 \leq a \leq 1000$ and $0 \leq b \leq 1000$. This means the extra factors have either $a = 1001$ or $b = 1001$.

If $a = 1001$, then there are 1001 options for b. If $b = 1001$ then there are 1001 options for a, which gives a total of 2002 factors. However, we have counted the factor $2^{1001} \times 5^{1001}$ twice, as it has $a = b = 1001$. This means there are $1001 + 1001 - 1 = 2001$ factors of the form that we want.

3. (a) Expand $(x + y)^2$ and $(x - y)^2$.

(b) Solve the following system of equations.

$$\begin{aligned} x^2 + xy + y^2 &= 21 \\ x^2 - xy + y^2 &= 13 \end{aligned}$$

Answer

(a)

$$\begin{aligned}(x+y)^2 &= x^2+2xy+y^2\\(x-y)^2 &= x^2-2xy+y^2\end{aligned}$$

(b) $x=4$ and $y=1$ or $x=1$ and $y=4$ or $x=-4$ and $y=-1$ or $x=-1$ and $y=-4$.

Solution

(a) $(x+y)^2 = x^2+xy+xy+y^2 = x^2+2xy+y^2$.

$(x-y)^2 = x^2-xy-xy+y^2 = x^2-2xy+y^2$.

(b) If we add the equations, then we get $2x^2+2y^2=34$, so $x^2+y^2=17$. Subtracting the second equation from the first gives $2xy=8$.

We can then add and subtract these new equations, which gives us the following:

$$\begin{aligned}x^2+2xy+y^2 &= 25;\\x^2-2xy+y^2 &= 9.\end{aligned}$$

Using the first part of this question, we can factorise to obtain the equations:

$$\begin{aligned}(x+y)^2 &= 25;\\(x-y)^2 &= 9.\end{aligned}$$

It follows that $x+y=\pm 5$, and $x-y=\pm 3$. This gives a total of four options. Solving the simultaneous equations in each case gives the possible pairs for (x, y) of $(4, 1)$, $(1, 4)$, $(-1, -4)$ and $(-4, -1)$.

4. If the digits "15" are inserted in the middle of the square number 16, then the number 1156, which is also a perfect square, is produced. If this process is repeated, then we obtain the numbers 111556, 11115556, 1111155556 and so on. Prove that each number in this sequence is a square.

Solution

> If we look at the first few terms of this sequence, then we can see that $16=4^2$, $1156=34^2$, $111556=334^2$ and so on. This suggests that we should look at these numbers.

Consider the calculation $33\ldots34^2$. Notice that $33\ldots34\times 3 = 100\ldots02$ and $33\ldots34\times 4 = 133\ldots36$.

Then we have the following long multiplication.

$$\begin{array}{ccccccccccc}
 & & & & 3 & 3 & \ldots & 3 & 3 & 4\\
 & & & \times & 3 & 3 & \ldots & 3 & 3 & 4\\\hline
 & & & 1 & 3 & 3 & \ldots & 3 & 3 & 6\\
 & & 1 & 0 & 0 & \ldots & 0 & 0 & 2 & 0\\
 & & \vdots & & & & & & & \vdots\\
1 & 0 & \ldots & 0 & 0 & 2 & 0 & 0 & \ldots & 0 & 0\\\hline
1 & 1 & \ldots & 1 & 1 & 5 & 5 & \ldots & 5 & 5 & 6\\\hline
\end{array}$$

We need to check that all the columns line up in the way that the calculation displayed suggests. Suppose that the number being squared has k digits. This means that there are $(k-1)$ 3's in the first row, appearing in the second to k^{th} columns from the right. There are then $k-1$ rows below, each of which puts a 2 in the second through to the k^{th} column from the right, and a 1 in the $(k+1)^{\text{st}}$ to $2k^{\text{th}}$ columns. These rows add to give k 1s, $k-1$ 5s and a 6, which is what we wanted.

This then means that we have the terms from the required sequence each time, so these terms are all square numbers.

5. (a) Suppose that a and b are non-negative numbers. Prove that $\sqrt{ab} \leq \dfrac{a+b}{2}$.

This result is called the *AM-GM* inequality, because it says that the geometric mean, $\sqrt{ab}$, is less than or equal to the arithmetic mean, $\dfrac{a+b}{2}$.

(b) Suppose $x > 0$. What is the minimum possible value of $x + \dfrac{1}{x}$?

(c) Suppose that x and y are positive numbers such that $x + y = 2$. Prove that $2xy\left(x^2 + y^2\right) \leq 4$.

Solution

(a)

It is not obvious where to start in this question, so let us try rearranging the inequality that we are trying to prove. If we square it, then we get

$$ab \leq \left(\frac{a+b}{2}\right)^2 = \frac{a^2 + 2ab + b^2}{4}.$$

Then, multiplying by 4 and subtracting $4ab$ from each side gives

$$0 \leq a^2 - 2ab + b^2 = (a-b)^2.$$

We know that this is true, since squares are larger than, or equal to, zero. The challenge is to reverse this into a proof.

We know that $(a-b)^2 \geq 0$, since this is true for all squares. Expanding this, we get that

$$0 \leq a^2 - 2ab + b^2.$$

Then we can add $4ab$ to each side to obtain

$$4ab \leq a^2 + 2ab + b^2.$$

Factorising the right hand side, and dividing by 4 yields

$$ab \leq \frac{(a+b)^2}{4}.$$

Now, since a and b are both non-negative, we can take the positive square root of both sides, giving

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

(b) We can apply the above result with $a = x$ and $b = \frac{1}{x}$. Then the AM-GM inequality tells us that

$$\sqrt{x \times \frac{1}{x}} \leq \frac{x + \frac{1}{x}}{2}.$$

Simplifying this, we get $x + \frac{1}{x} \geq 2$. We can obtain this when $x = 1$, so this really is the minimum.

(c)

> We know the value of $x + y$, so we want to express things that we know in terms of $x + y$. One example of this is $(x + y)^2 = x^2 + y^2 + 2xy$. What is more, the expression we are trying to prove something about is the product of two parts of this formula.

Let us apply the AM-GM inequality to $a = 2xy$ and $b = x^2 + y^2$. This tells us that

$$\sqrt{2xy(x^2 + y^2)} \leq \frac{x^2 + y^2 + 2xy}{2} = \frac{(x + y)^2}{2}.$$

Square this and use the condition $x + y = 2$. This produces

$$2xy\left(x^2 + y^2\right) \leq \left(\frac{2^2}{2}\right)^2 = 2^2 = 4.$$

This is precisely what we wanted to prove.

6. Each element of the set $\{1, 2, \ldots, n\}$ is to be coloured red or blue in such a way that no four numbers of the same colour (not necessarily distinct) satisfy $w + x + y = z$. Determine the largest positive integer n for which is can be achieved.

ANSWER 10

SOLUTION

> It takes some inspiration to try this with the right number and then see if it works. One suggestion is to try without a limit, and then see what number causes you a problem. What you need is a number n that works, but where $n + 1$ does not work.

Suppose we consider the case where $n = 10$. We can assume that 1 is coloured blue - if it is red then we can swap all the colours over.

Now, if we have a sum $a + b + c = d$ where all the numbers except one are the same colour, then the other number must be of the other colour. We can now use this fact to try and colour the other numbers.

As $1 + 1 + 1 = 3$ and 1 is blue, this means that 3 is red. Likewise, as $3 + 3 + 3 = 9$ and 3 is red, this means 9 is blue.

Blue:	1	9
Red:	3	

Then we know that $1 + 1 + 7 = 9$, and 1 and 9 are blue, so 7 must be red. Again, $2 + 2 + 3 = 7$, and 3 and 7 are red, so 2 is blue. We can tabulate these results as follows.

Blue:	1	2	9			
Red:	3	7				

Now, 1 and 2 are both blue, and $1 + 1 + 2 = 4$, $1 + 2 + 2 = 5$, $2 + 2 + 2 = 6$. This means that 4, 5 and 6 must all be red, so we have the following table.

Blue:	1	2	9			
Red:	3	4	5	6	7	

Then, as $3 + 3 + 4 = 10$ and 3 and 4 are both red, 10 must be blue. However, we have $1 + 1 + 8 = 10$ with 1 and 10 both blue, so 8 must be red. This produces the following table.

Blue:	1	2	9	10		
Red:	3	4	5	6	7	8

Now, with this split, adding three of the red numbers gives at least $3 + 3 + 3 = 9 > 8$, so there are no sums amongst the red numbers. Since $1 + 1 + 1 > 2$, $2 + 2 + 2 < 9$ and $1 + 1 + 9 > 10$, there are no other possibilities. Therefore this colouring of the numbers 1 to 10 satisfies the given condition.

However, it is now straightforward to show that it is the only possibility. Consider the case $n = 11$: we know the colours of the first ten numbers - they must be as they were above. But then, $1+1+9 = 11$, so 11 cannot be blue. However, $3 + 4 + 4 = 11$, so 11 cannot be red. This means we cannot colour 11 numbers.

Since we can't colour 11 numbers, we also cannot colour more than 11 numbers, as this would involve colouring 11 numbers as part of it. Therefore the maximum value of n is 10.

7. Ten girls numbered from 1 to 10 are seated around a circular table. Each girl writes down the sum of her own number and those of the two people next to her.

(a) Prove that at least one of the girls writes down a number bigger than 16.

(b) Prove that at least one of the girls writes down a number bigger than 17.

Solution

(a) We can consider the total of the numbers written down by all the girls. Each girl's number contributes to exactly three of the numbers written - that girl and the two on either side. This means that the total number written down is $3 \times (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) = 3 \times 55 = 165$.

This implies that the average number written down must be $165 \div 10 = 16.5$, so at least one girl must write down a number greater than or equal to 16.5. Hence certainly one of the girls must have written down a number that is at least 17.

(b)

> In the previous part, we used the fact that some girl wrote down a number greater than the average. For this part, we want to find a subset of girls whose numbers must have a higher average.

One of the girls has the number 1. Working round from her, we can denote the arrangement by

$$1, a, b, c, d, e, f, g, h, i.$$

Now consider the girls with numbers b, e and h. These girls write down totals of $a + b + c$, $d + e + f$ and $g + h + i$. The total of these must then be the total of all the numbers between 2 and 10, so is $2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 54$. The average of the three sums is $54 \div 3 = 18$. This means that at least one of these girls must have written down a number that is at least 18.